

Optimal control on distributions

Constantin Udriște

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This paper studies (single-time and multitime) optimal control problems on a nonholonomic manifold (described either by the kernel of a Gibbs-Pfaff form or by the span of appropriate vector fields). For both descriptions we analyse: infinitesimal deformations and adjointness, single-time optimal control problems, multitime optimal control problem of maximizing a multiple integral functional, multitime optimal control problem of maximizing a curvilinear integral functional, Curvilinear functionals depending on curves, optimization of mechanical work on Riemannian manifolds. Also we prove that a nonholonomic system can be always controlled by uni-temporal or bi-temporal bang-bang controls.

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1 Optimal control on a distribution described by a Pfaff equation

A *generalized distribution* $\Delta : x \rightarrow \Delta_x$, or Stefan-Sussmann distribution, is similar to a distribution, but the subspaces are not required to be all of the same dimension. The definition requires that the subspaces Δ_x are locally spanned by a set of vector fields, but these will no longer be everywhere linearly independent. It is not hard to see that the dimension of the distribution Δ is lower semicontinuous, so that at special points the dimension is lower than at nearby points. One class of examples is provided by a non-free action of a Lie group on a manifold, the vector fields in question being the

infinitesimal generators of the group action (a free action gives rise to a genuine distribution). Another examples arise in dynamical systems, where the set of vector fields in the definition is the set of vector fields that commute with a given one. There are also examples and applications in Control theory, where the generalized distribution represents infinitesimal constraints of the system.

see ControlJakubczyk, pag 146

Lemma *If the variational system (treated as linear system without constraints on the control) is controllable, then the original system is strongly accessible.*

Nonholonomic path planning represents a fusion of some of the newest ideas in control theory, classical mechanics, and differential geometry with some of the most challenging practical problems in robot motion planning. Furthermore, the class of systems to which the theory is relevant is broad: mobile robots, space-based robots, multifingered hands, and even such systems as a one-legged hopping robot. The techniques presented here indicate one possible method for generating efficient and computable trajectories for some of these nonholonomic systems in the absence of obstacles.

The delay differential equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. For example

$$\frac{d}{dt}x(t) = f(x(t), x(t - \tau)).$$

A delay Pfaff equation means

$$a_i(x(t), x(t - \tau))dx^i(t) = 0.$$

1.1 Infinitesimal deformations and adjointness on distributions

Let D be a nonholonomic distribution on R^n described by a Pfaff equation

$$(1) \quad a_i(x)dx^i = 0.$$

Let $x(t)$, $t \in I = [t_0, t_1] \subset R$, be an integral curve of the distribution D . Let $x(t; \epsilon)$, $\epsilon \in [0, \delta)$ be a differentiable variation of $x(t)$, i.e.,

$$A_i(x(t; \epsilon); \epsilon)dx^i(t; \epsilon) = 0, A_i(x; 0) = a_i(x), x(t; 0) = x(t).$$

The variation $(t, \epsilon) \rightarrow x(t; \epsilon)$, $t \in I$, $\epsilon \in [0, \delta)$ is a surface. It is an integral surface only if the distribution D admits integral surfaces. Taking the partial derivative with respect to ϵ and denoting $y^i(t) = \frac{\partial x^i}{\partial \epsilon}(t; 0)$, we find the *single-time (Pfaff) infinitesimal deformation equation*

$$(2) \quad \left(\frac{\partial a_i}{\partial x^j}(x) y^j(t) + b_i(x) \right) dx^i + a_i(x) dy^i = 0, \quad b_i(x) = \frac{\partial A_i}{\partial \epsilon}(x; 0)$$

around a solution $x(t)$ of the Pfaff equation (1). The *single-time adjoint Pfaff system* is

$$(3) \quad d(p(t) a_j(x(t))) = \frac{\partial(p(t) a_i(x) dx^i)}{\partial x^j},$$

whose solution $p(t)$ is called the *costate function*. The foregoing Pfaff equations (2) and (3) are *adjoint (dual)* in the following sense: if y is a solution of the infinitesimal deformation (Pfaff) equation (2), then the function $p(t) a_i(x) y^i(t)$ verifies the Pfaff equation $d(p(t) a_i(x) y^i(t)) + p(t) b_i(x) dx^i = 0$.

Let $x(t)$, $t \in \Omega_{t_0 t_1} \subset R_+^m$ be a maximal, m -dimensional, $m \leq n - 2$, integral submanifold of the distribution D . We fix $k = 1, \dots, n - m - 1$. Let $\epsilon = (\epsilon^A) \in [0, \delta)^k$, $A = 1, \dots, k$, and let $x(t; \epsilon)$ be a differentiable variation of $x(t)$, i.e.,

$$A_i(x(t; \epsilon); \epsilon) dx^i(t; \epsilon) = 0, \quad A_i(x, 0) = a_i(x), \quad x(t; 0) = x(t).$$

The variation $(t, \epsilon) \rightarrow x(t; \epsilon)$, $t \in \Omega_{t_0 t_1}$, $\epsilon \in [0, \delta)^k$ is an $(m + k)$ -dimensional manifold, but not an integral submanifold.

Taking the partial derivative with respect to ϵ^A and denoting $y_A^i(t) = \frac{\partial x^i}{\partial \epsilon^A}(t; 0)$, we find the *multitime infinitesimal deformation (Pfaff) system*

$$(4) \quad \left(\frac{\partial a_i}{\partial x^j}(x) y_A^j(t) + b_{iA}(x) \right) dx^i + a_i(x) dy_A^i = 0, \quad b_{iA}(x) = \frac{\partial A_i}{\partial \epsilon^A}(x; 0)$$

around a solution $x(t)$ of the Pfaff equation (1). The *multitime adjoint Pfaff system* is

$$(5) \quad d(p^A(t) a_j(x)) = \frac{\partial(p^A(t) a_i(x) dx^i)}{\partial x^j},$$

whose solution $p(t) = (p^A(t))$ is called the *costate vector*. The foregoing Pfaff equations (4) and (5) are *adjoint (dual)* in the following sense: if y_A^i is a solution of the infinitesimal deformation (Pfaff) system (4), then the function $a_i(x)p^A(t)y_A^i(t)$ verify the Pfaff equation $d(a_i(x)p^A(t)y_A^i(t)) + p^A(t)b_{iA}(x)dx^i = 0$.

1.2 Evolution of a distribution

Let D be a nonholonomic distribution on R^n described by a Pfaff equation (1). Let $x(t)$, $t \in I = [t_0, t_1] \subset R$, be an integral curve of the distribution D . Let $x(t; \epsilon)$, $\epsilon \in [0, \delta)$ be a differentiable variation of $x(t)$. Suppose that $(x(t; \epsilon); \epsilon)$ is an integral surface of a Pfaff equation in R^{n+1} , i.e.,

$$A_i(x(t; \epsilon), \epsilon)dx^i(t; \epsilon) = B(x(t; \epsilon), \epsilon)d\epsilon,$$

$$A_i(x, 0) = a_i(x), B(x, \epsilon) = A_i(x, \epsilon)\frac{\partial x^i}{\partial \epsilon}, B(x, 0) = 0, x(t; 0) = x(t).$$

Taking the partial derivative with respect to ϵ , we find

$$\left(\frac{\partial A_i}{\partial x^j} \frac{\partial x^j}{\partial \epsilon} + \frac{\partial A_i}{\partial \epsilon} \right) dx^i + A_i d \frac{\partial x^i}{\partial \epsilon} = \left(\frac{\partial B}{\partial x^j} \frac{\partial x^j}{\partial \epsilon} + \frac{\partial B}{\partial \epsilon} \right) d\epsilon.$$

If we accept an evolution after the direction of the vector field X^j , i.e., $\frac{\partial x^j}{\partial \epsilon} = \alpha X^j$, then we find the PDE system

$$\left(\frac{\partial A_i}{\partial x^j} X^j + \frac{\partial A_i}{\partial \epsilon} \right) dx^i + A_i dX^i = \left(\frac{\partial B}{\partial x^j} X^j + \frac{\partial B}{\partial \epsilon} \right) d\epsilon$$

with unknowns A_i , fixed by initial conditions. For $\epsilon = 0$, we rediscover the system in variations, with the condition $a_i(x)y^i(t) = 0$, $y^i(t) = \frac{\partial x^i}{\partial \epsilon}(t, 0)$.

1.3 Single-time optimal control problems on a distribution

Let D be a distribution on R^n described by a controlled Pfaff equation

$$a_i(x, u)dx^i = 0, x = (x^i) \in R^n, u = (u^a) \in R^k$$

and let $x(t)$, $t \in I = [t_0, t_1]$, be an integral curve of the distribution D .

A *single-time optimal control problem* consists of maximizing the functional

$$(6) \quad I(u(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), u(t)) dt + g(x(t_1))$$

subject to

$$(7) \quad a_i(x(t), u(t)) dx^i(t) = 0, \text{ a.e. } t \in I = [t_0, t_1], \quad x(t_0) = x_0.$$

It is supposed that $L : I \times A \times U \rightarrow R$ is a C^2 function, $a_i : A \times U \rightarrow R$, $i = 1, \dots, n$ are C^2 functions and $g(x(t_1))$ is a C^1 function. Ingredients: A is a bounded and closed subset of R^n which contains each trajectory $x(t)$, $t \in I$ of controlled system, and x_0 and x_1 are the initial and final states of the trajectory $x(t)$. The values of the control functions belong to a set $U \subset R^k$, bounded and closed.

Let us find the first order necessary conditions for an optimal pair (x, u) . We fix the control $u(t)$ and we variate the state $x(t)$ into $x(t, \epsilon)$. We obtain the *single-time infinitesimal deformation (Pfaff) equation*

$$\frac{\partial a_i}{\partial x^j}(x, u) y^j dx^i + a_i(x, u) dy^i = 0.$$

of the nonholonomic constraint $a_i(x(t), u(t)) dx^i(t) = 0$. It follows the *single-time adjoint Pfaff equation*

$$d(p(t) a_j(x(t), u(t))) = \frac{\partial(p(t) a_i(x, u) dx^i)}{\partial x^j},$$

whose solution $p(t)$ is called the *costate function*. Here, the symbol d in the left hand member of the adjoint equation means the differentiation with respect to p and x .

Using the Lagrangian 1-form

$$\mathcal{L} = L(t, x(t), u(t)) dt,$$

we build the *Hamiltonian 1-form*

$$\mathcal{H} = \mathcal{L}(t, x(t), u(t)) + p(t) a_i(x(t), u(t)) dx^i(t).$$

Theorem (Single-time maximum principle) *Suppose that the problem of maximizing the functional (6) constrained by (7) has an interior optimal solution $\hat{u}(t)$, which determines the optimal evolution $x(t)$. Then there exists a costate function $(p(t))$ such that*

$$(8) \quad \frac{\partial \mathcal{H}}{\partial p} = a_i(x(t), u(t)) dx^i(t) = 0,$$

the function $(p(t))$ is the unique solution of the following Pfaff system (adjoint system)

$$(9) \quad d(p(t)a_j(x(t), u(t))) = \frac{\partial \mathcal{H}}{\partial x^j}$$

and satisfies the critical point conditions

$$(10) \quad \mathcal{H}_{u^a}(t, x(t), u(t), p(t)) = 0, \quad a = \overline{1, k}.$$

Proof We use the Hamiltonian 1-form \mathcal{H} . The solutions of the foregoing problem are among the solutions of the free maximization problem of the curvilinear integral functional

$$J(u(\cdot)) = \int_{\tilde{\Gamma}} \mathcal{H}(t, x(t), u(t), p(T)) + g(x(t_1)),$$

where $\tilde{\Gamma} = ([t_0, t_1], x([t_0, t_1])) = \{(t, x(t)) | t \in [t_0, t_1]\} \subset R_+ \times R^n$.

Suppose that there exists a continuous control $\hat{u}(t)$, $t \in I = [t_0, t_1]$, with $\hat{u}(t) \in \text{Int } U$, and an integral curve $x(t)$ which are optimal in the previous problem. Now consider a control variation $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t)$, $t \in I = [t_0, t_1]$, where h is an arbitrary continuous vector function, and a state variation $x(t, \epsilon)$, $t \in I = [t_0, t_1]$, related by

$$a_i(x(t; \epsilon), u(t; \epsilon)) dx^i(t; \epsilon) = 0, \quad x(t; 0) = x(t), \quad x(t_0, 0) = x_0.$$

Since $\hat{u}(t) \in \text{Int } \mathcal{U}$ and any continuous function over a compact set I is bounded, there exists $\epsilon_h > 0$ such that $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t) \in \text{Int } U$, $\forall |\epsilon| < \epsilon_h$. This ϵ is used in our variational arguments.

For $|\epsilon| < \epsilon_h$, we define the function

$$J(\epsilon) = \int_{\tilde{\Gamma}(\epsilon)} \mathcal{H}(t, x(t, \epsilon), u(t, \epsilon), p(t)) + g(x(t_1, \epsilon))$$

$$= \int_{\tilde{\Gamma}(\epsilon)} \mathcal{L}(t, x(t, \epsilon), u(t, \epsilon)) + p(t) a_i(x(t, \epsilon), u(t, \epsilon)) dx^i(t, \epsilon) + g(x(t_1, \epsilon)).$$

Differentiating with respect to ϵ , it follows

$$\begin{aligned} J'(\epsilon) &= \int_{\tilde{\Gamma}(\epsilon)} \mathcal{L}_{x^j}(t, x(t, \epsilon), u(t, \epsilon)) x_\epsilon^j(t, \epsilon) + \frac{\partial g}{\partial x^j}(x(t_1, \epsilon)) x_\epsilon^j(t_1, \epsilon) \\ &\quad + \int_{\tilde{\Gamma}(\epsilon)} p(t) \frac{\partial a_i}{\partial x^j}(x(t, \epsilon), u(t, \epsilon)) x_\epsilon^j(t, \epsilon) dx^i(t, \epsilon) \\ &\quad + \int_{\tilde{\Gamma}(\epsilon)} p(t) a_i(x(t, \epsilon), u(t, \epsilon)) dx_\epsilon^i(t, \epsilon) \\ &\quad + \int_{\tilde{\Gamma}(\epsilon)} \mathcal{L}_{u^a}(t, x(t, \epsilon), u(t, \epsilon)) h^a(t) \\ &\quad + \int_{\tilde{\Gamma}(\epsilon)} p(t) \frac{\partial a_i}{\partial u^a}(x(t, \epsilon), u(t, \epsilon)) h^a(t) dx^i(t, \epsilon). \end{aligned}$$

Evaluating at $\epsilon = 0$, we find $\tilde{\Gamma}(0) = \tilde{\Gamma}$ and

$$\begin{aligned} J'(0) &= \int_{\tilde{\Gamma}} \left(\mathcal{L}_{x^j}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial x^j}(x(t), \hat{u}(t)) dx^i(t) \right) x_\epsilon^j(t, 0) \\ &\quad + \int_{\tilde{\Gamma}} p(t) a_i(x(t), \hat{u}(t)) dx_\epsilon^i(t, 0) + \frac{\partial g}{\partial x^j}(x(t_1, 0)) x_\epsilon^j(t_1, 0) \\ &\quad + \int_{\tilde{\Gamma}} \left(\mathcal{L}_{u^a}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial u^a}(x(t), \hat{u}(t)) dx^i(t) \right) h^a(t), \end{aligned}$$

where $x(t)$ is the curve of the state variable corresponding to the optimal control $\hat{u}(t)$. The integral from the middle can be written

$$\int_{\tilde{\Gamma}} p(t) a_j(x(t), \hat{u}(t)) dx_\epsilon^j(t, 0) = p(t) a_j x_\epsilon^j|_{\partial \tilde{\Gamma}} - \int_{\tilde{\Gamma}} d(p(t) a_j(x(t), \hat{u}(t))) x_\epsilon^j(t, 0),$$

where the symbol d in the last integral means the differentiation with respect to p and x . We find $J'(0)$ as

$$\begin{aligned} &\int_{\tilde{\Gamma}} \left(\mathcal{L}_{x^j}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial x^j}(x(t), \hat{u}(t)) dx^i(t) - d(p(t) a_j(x(t), \hat{u}(t))) \right) x_\epsilon^j(t, 0) \\ &\quad + p(t) a_j(x(t), \hat{u}(t)) x_\epsilon^j(t, 0)|_{t_0}^{t_1} + \frac{\partial g}{\partial x^j}(x(t_1, 0)) x_\epsilon^j(t_1, 0) \end{aligned}$$

$$+ \int_{\bar{\Gamma}} \left(\mathcal{L}_{u^a}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial u^a}(x(t), \hat{u}(t)) dx^i(t) \right) h^a(t).$$

Since $x(t_0) = x_0$, we have $x_\epsilon^j(t_0, 0) = 0$. We select the costate function $p(t)$ as solution of the adjoint Pfaff equation

$$\mathcal{L}_{x^j}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial x^j}(x(t), \hat{u}(t)) dx^i(t) - d(p(t) a_j(x(t), \hat{u}(t))) = 0,$$

with the terminal condition $p(t_1) a_j(x(t_1), \hat{u}(t_1)) + \frac{\partial g}{\partial x^j}(x(t_1)) = 0$. On the other hand, we need $J'(0) = 0$ for all $h(t) = (h^a(t))$. Since the variation h is arbitrary, we get the following (*critical point condition*)

$$\frac{\partial \mathcal{L}}{\partial u^a}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial u^a}(x(t), \hat{u}(t)) dx^i(t) = 0.$$

The foregoing equations (9) and (10) can be written

$$d(p(t) a_j(x(t), u(t))) = \frac{\partial \mathcal{H}}{\partial x^j}(t, x(t), u(t), p(t)),$$

$$\frac{\partial \mathcal{H}}{\partial u^a}(t, x(t), u(t), p(t)) = 0.$$

Example Let us solve the problem

$$\max J(u(\cdot)) = -\frac{1}{2} \int_{t_0}^{t_1} (u^2(t) + z^2(t)) dt$$

subject to (controlled Martinet distribution)

$$dz = \frac{1}{2}(y^2 + u)dx, \quad x(t_0) = x_0, y(t_0) = y_0, z(t_0) = z_0.$$

Denote $\omega = \frac{1}{2}(y^2 + u)dx - dz$. Since

$$\omega \wedge d\omega = \frac{1}{2}dx \wedge d(y^2 + u) \wedge dz$$

the distribution admits only integral curves with the parameter t . The Pfaff equation

$$dz = \frac{1}{2}(y^2 + u)dx,$$

is equivalent to a differential equation

$$\dot{z}(t) = \frac{1}{2}(y(t)^2 + u(t))\dot{x}(t)$$

or to the ODE system

$$\dot{x}(t) = \varphi(t), \dot{y}(t) = \psi(t), \dot{z}(t) = \frac{1}{2}(y(t)^2 + u(t))\varphi(t).$$

First variant Using the Hamiltonian

$$H = -\frac{1}{2}(u^2(t) + z^2(t)) + p_1(t)\varphi(t) + p_2(t)\psi(t) + p_3(t) \left(\frac{1}{2}(y(t)^2 + u(t))\varphi(t) \right),$$

we find the adjoint ODEs

$$\dot{p}_1(t) = -\frac{\partial H}{\partial x} = 0, \dot{p}_2(t) = -\frac{\partial H}{\partial y} = -p_3 y \varphi, \dot{p}_3(t) = -\frac{\partial H}{\partial z} = z(t)$$

and the critical point condition

$$\frac{\partial H}{\partial u} = -u + \frac{1}{2} p_3 \varphi = 0.$$

We find the control $u = \frac{1}{2} p_3 \varphi$ and $p_1(t) = c_1$. We need to find solutions for the system

$$\dot{p}_2(t) = -p_3(t)y(t)\dot{x}(t), \dot{p}_3(t) = z(t), \dot{z}(t) = \frac{1}{2} \left(y(t)^2 + \frac{1}{2} p_3(t)\dot{x}(t) \right) \dot{x}(t).$$

Second variant The Hamiltonian 1-form is

$$\mathcal{H} = -\frac{1}{2}(u^2(t) + z^2(t))dt + p(t) \left(\frac{1}{2}(y^2 + u)dx - dz \right).$$

The critical point condition

$$\frac{\partial \mathcal{H}}{\partial u} = -u dt + \frac{1}{2} p dx = 0$$

gives the control $u = \frac{1}{2} p \dot{x}$.

Since $a_1 = \frac{1}{2}(y^2 + u)$, $a_2 = 0$, $a_3 = -1$, we find the adjoint Pfaff equations

$$d(p \frac{1}{2}(y^2 + u)) = \frac{\partial \mathcal{H}}{\partial x} = 0, 0 = \frac{\partial \mathcal{H}}{\partial y} = p y dx, dp = -\frac{\partial \mathcal{H}}{\partial z} = z dt.$$

We need to solve the system

$$\begin{aligned} p(y^2 + u) &= c, pydx = 0, dp = zdt, \\ udt &= \frac{1}{2}pdx, dz = \frac{1}{2}(y^2 + u)dx. \end{aligned}$$

On the other hand, the second variant offers two explicit extremals: (1) $y = 0, pu = c_1, dp = zdt, udt = \frac{1}{2}pdx, dz = udx$, which does not satisfy the general initial conditions; (2) $x = c_1, u = 0, z = c_3, p = c_3t + c_4, y = \pm\sqrt{\frac{c}{c_3t+c_4}}$, depending upon four arbitrary constants, which determine from initial conditions and terminal condition.

The first variant can be identified with the second variant via $p_1 = pa_1; p_2 = pa_2, p_3 = -pa_3$.

Third variant (Ionel Tevy) We introduce two auxiliary controls u_1, u_2 , changing the Pfaff equation into the controlled ODE system

$$\dot{x}(t) = u_1(t), \dot{y}(t) = u_2(t), \dot{z}(t) = \frac{1}{2}(y(t)^2 + u(t))u_1(t).$$

Then

$$H = -\frac{1}{2}(z^2 + u^2) + p_1u_1 + p_2u_2 + \frac{1}{2}p(y^2 + u)u_1.$$

We find the adjoint ODEs and the critical point conditions

$$\dot{p}_1 = -\frac{\partial H}{\partial x} = 0, \dot{p}_2 = -\frac{\partial H}{\partial y} = -pyu_1, \dot{p} = z,$$

$$\frac{\partial H}{\partial u_1} = p_1 + \frac{1}{2}p(y^2 + u) = 0, \frac{\partial H}{\partial u_2} = p_2 = 0, \frac{\partial H}{\partial u} = -u + \frac{1}{2}pu_1 = 0.$$

It follows two extremals:

$$\begin{aligned} p_1(t) &= c, p_2(t) = 0, p(t) = -c_3t - a, \\ u_1(t) &= 0, u_2(t) = \dot{y}(t), u(t) = 0, \\ x(t) &= c_1, y(t) = \pm\sqrt{\frac{2c}{c_3t+a}}, z = -c_3; \\ p_1(t) &= c, p_2(t) = 0, p(t) = \sqrt{\frac{(\alpha t + \beta)^2 + 4c^2}{\alpha}}, \end{aligned}$$

$$u_1 = \frac{4\alpha c}{(\alpha t + \beta)^2 + 4c^2}, u_2(t) = 0, u(t) = -2c\sqrt{\frac{\alpha}{(\alpha t + \beta)^2 + 4c^2}},$$

$$x(t) = \int u_1(t)dt = 2 \operatorname{atan}\frac{\alpha t + \beta}{2c} + c_1, y(t) = 0, z(t) = \frac{\alpha t + \beta}{\sqrt{(\alpha t + \beta)^2 + 4c^2}}.$$

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1.4 Multitime optimal control problems on a distribution

Let D be a distribution on R^n described by a controlled Pfaff equation

$$a_i(x, u)dx^i = 0, x = (x^i) \in R^n, u = (u^a) \in R^k$$

and let $x(t)$, $t \in \Omega_{t_0 t_1} \subset R^m$ be an m -dimensional ($m < n$, m maximal) integral submanifold of the distribution D .

Let us start with a

1.4.1 Multitime optimal control problem of maximizing a multiple integral functional

Find

$$(11) \quad \max_{u(\cdot)} I(u(\cdot)) = \int_{\Omega_{t_0 t_1}} L(t, x(t), u(t)) \omega + \int_{\partial \Omega_{t_0 t_1}} g(x(t)) d\sigma$$

subject to

$$(12) \quad a_i(x(t), u(t))dx^i(t) = 0, \text{ a.e. } t \in \Omega_{t_0 t_1}, x(t_0) = x_0.$$

It is supposed that $L : \Omega_{t_0 t_1} \times A \times U \rightarrow R$ is a C^2 function and $a_i : A \times U \rightarrow R$, $i = 1, \dots, n$ are C^2 functions. Ingredients: $\omega = dt^1 \cdots dt^m$ is the volume element, A is a bounded and closed subset of R^n , containing the images of the m -sheets $x(t)$, $t \in \Omega_{t_0 t_1}$ of controlled system, and x_0 and x_1 are the initial and final states of an m -sheet $x(t)$. The set in which the control functions takes their values is called as U , which is a bounded and closed subset of R^k .

Let us find the first order necessary conditions for an optimal pair (x, u) . We fix the control $u(t)$ and variate the state $x(t)$ into $x(t, \epsilon)$, $\epsilon = (\epsilon^1, \dots, \epsilon^m)$. We find the *multitime infinitesimal deformation (Pfaff) system*

$$\frac{\partial a_i}{\partial x^j}(x, u)y_\alpha^j(t)dx^i + a_i(x, u)dy_\alpha^i = 0$$

of the nonholonomic constraint $a_i(x(t), u(t))dx^i(t) = 0$. It follows the *multitime adjoint Pfaff system*

$$d(p^\alpha(t)a_j(x, u)) = \frac{\partial(p^\alpha(t)a_i(x, u)dx^i)}{\partial x^j},$$

whose solution $p(t) = (p^\alpha(t))$ is called the *costate vector*. Here, the symbol d in the left hand member of the adjoint equation means the differentiation with respect to p and x .

We use the Lagrangian m -form

$$\mathcal{L} = L(t, x(t), u(t))\omega.$$

Introducing the $(m-1)$ -forms

$$\omega_\lambda = \frac{\partial}{\partial t^\lambda} \rfloor \omega,$$

a *costate variable vector* or *Lagrange multiplier vector* $p = p^\alpha(t) \frac{\partial}{\partial t^\alpha}$ is identified to the $(m-1)$ -form $p = p^\lambda(t)\omega_\lambda$. We build a *Hamiltonian m-form*

$$\mathcal{H} = \mathcal{L}(t, x(t), u(t)) + p^\lambda(t)a_i(x(t), u(t))dx^i(t) \wedge \omega_\lambda.$$

Theorem (Multitime maximum principle) *Suppose that the problem of maximizing the functional (11) constrained by (12) has an interior optimal solution $\hat{u}(t)$, which determines the optimal evolution $x(t)$. Then there exists a costate function $(p(t))$ such that*

$$(13) \quad \frac{\partial \mathcal{H}}{\partial p^\lambda} = a_i(x(t), u(t))dx^i(t) \wedge \omega_\lambda = 0,$$

the function $(p(t))$ is the unique solution of the following Pfaff system (adjoint system)

$$(14) \quad d(p^\lambda a_j(x, u)) \wedge \omega_\lambda = \frac{\partial \mathcal{H}}{\partial x^j}$$

and satisfies the critical point conditions

$$(15) \quad \mathcal{H}_{u^a}(t, x(t), u(t), p(t)) = 0, \quad a = \overline{1, k}.$$

Proof We use the Hamiltonian m -form \mathcal{H} . The solutions of the foregoing problem are among the solutions of the free maximization problem of the functional

$$J(u(\cdot)) = \int_{\tilde{\Omega}} \mathcal{H}(t, x(t), u(t), p(t)) + \int_{\partial \tilde{\Omega}} g(x(t)) d\sigma$$

where $\tilde{\Omega} = (\Omega_{t_0 t_1}, x(\Omega_{t_0 t_1})) \subset R_+^m \times R^n$.

Suppose that there exists a continuous control $\hat{u}(t)$ defined over the interval $\Omega_{t_0 t_1}$ with $\hat{u}(t) \in \text{Int } U$ which is an optimum point in the previous problem. Now we consider a control variation $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t)$, where h is an arbitrary continuous vector function, and a state variation $x(t, \epsilon)$, $t \in \Omega_{t_0 t_1}$, connected by

$$a_i(x(t, \epsilon), u(t, \epsilon)) dx^i(t, \epsilon) = 0, \quad x(t, 0) = x(t), \quad x(t_0, 0) = x_0.$$

Since $\hat{u}(t) \in \text{Int } U$ and any continuous function over a compact set $\Omega_{t_0 t_1}$ is bounded, there exists $\epsilon_h > 0$ such that $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t) \in \text{Int } U$, $\forall |\epsilon| < \epsilon_h$. This ϵ is used in our variational arguments.

For $|\epsilon| < \epsilon_h$, we define the function

$$\begin{aligned} J(\epsilon) &= \int_{\tilde{\Omega}(\epsilon)} \mathcal{H}(t, x(t, \epsilon), u(t, \epsilon), p(t)) + \int_{\partial \tilde{\Omega}(\epsilon)} g(x(t, \epsilon)) d\sigma \\ &= \int_{\tilde{\Omega}(\epsilon)} \mathcal{L}(t, x(t, \epsilon), u(t, \epsilon)) + p^\lambda(t) a_i(x(t, \epsilon), u(t, \epsilon)) dx^i(t, \epsilon) \wedge \omega_\lambda + \int_{\partial \tilde{\Omega}(\epsilon)} g(x(t, \epsilon)) d\sigma. \end{aligned}$$

Differentiating with respect to ϵ , it follows

$$\begin{aligned} J'(\epsilon) &= \int_{\tilde{\Omega}(\epsilon)} \mathcal{L}_{x^j}(t, x(t, \epsilon), u(t, \epsilon)) x_\epsilon^j(t, \epsilon) + \int_{\partial \tilde{\Omega}(\epsilon)} g_{x^j}(x(t, \epsilon)) x_\epsilon^j(t, \epsilon) d\sigma \\ &\quad + \int_{\tilde{\Omega}(\epsilon)} p^\lambda(t) \frac{\partial a_i}{\partial x^j}(x(t, \epsilon), u(t, \epsilon)) x_\epsilon^j(t, \epsilon) dx^i(t, \epsilon) \wedge \omega_\lambda \\ &\quad + \int_{\tilde{\Omega}(\epsilon)} p^\lambda(t) a_i(x(t, \epsilon), u(t, \epsilon)) dx_\epsilon^i(t, \epsilon) \wedge \omega_\lambda \end{aligned}$$

$$\begin{aligned}
& + \int_{\tilde{\Omega}(\epsilon)} \mathcal{L}_{u^a}(t, x(t, \epsilon), u(t, \epsilon)) h^a(t) \\
& + \int_{\tilde{\Omega}(\epsilon)} p^\lambda(t) \frac{\partial a_i}{\partial u^a}(x(t, \epsilon), u(t, \epsilon)) h^a(t) dx^i(t, \epsilon) \wedge \omega_\lambda.
\end{aligned}$$

Evaluating at $\epsilon = 0$, we find $\tilde{\Omega}(0) = \tilde{\Omega}$ and

$$\begin{aligned}
J'(0) &= \int_{\tilde{\Omega}} \left(\mathcal{L}_{x^j}(t, x(t), \hat{u}(t)) + p^\lambda(t) \frac{\partial a_i}{\partial x^j}(x(t), \hat{u}(t)) dx^i(t) \wedge \omega_\lambda \right) x_\epsilon^j(t, 0) \\
&+ \int_{\tilde{\Omega}} p^\lambda(t) a_i(x(t), \hat{u}(t)) dx_\epsilon^i(t, 0) \wedge \omega_\lambda + \int_{\partial \tilde{\Omega}} g_{x^j}(x(t)) x_\epsilon^j(t, 0) d\sigma \\
&+ \int_{\tilde{\Omega}} \left(\mathcal{L}_{u^a}(t, x(t), \hat{u}(t)) + p^\lambda(t) \frac{\partial a_i}{\partial u^a}(x(t), \hat{u}(t)) dx^i(t) \wedge \omega_\lambda \right) h^a(t),
\end{aligned}$$

where $x(t)$ is the m -sheet of the state variable corresponding to the optimal control $\hat{u}(t)$.

To evaluate the multiple integral

$$\int_{\tilde{\Omega}} p^\lambda(t) a_i(x(t), \hat{u}(t)) dx_\epsilon^i(t, 0) \wedge \omega_\lambda,$$

we integrate by parts, via the formula

$$d(p^\lambda a_i x_\epsilon^i \omega_\lambda) = (x_\epsilon^i d(p^\lambda a_i) + p^\lambda a_i dx_\epsilon^i) \wedge \omega_\lambda,$$

obtaining

$$\int_{\tilde{\Omega}} p^\lambda a_i dx_\epsilon^i \wedge \omega_\lambda = \int_{\tilde{\Omega}} d(p^\lambda a_i x_\epsilon^i \omega_\lambda) - \int_{\tilde{\Omega}} d(p^\lambda a_i) x_\epsilon^i \wedge \omega_\lambda,$$

where the symbol d in the last integral means the differentiation with respect to p and x . Now we apply the Stokes integral formula

$$\int_{\tilde{\Omega}} d(p^\lambda a_i x_\epsilon^i \omega_\lambda) = \int_{\partial \tilde{\Omega}} \delta_{\alpha\beta} p^\alpha a_i x_\epsilon^i n^\beta d\sigma,$$

where $(n^\beta(t))$ is the unit normal vector to the boundary $\partial \tilde{\Omega}$. Since the integral from the middle can be written

$$\int_{\tilde{\Omega}} p^\lambda a_i dx_\epsilon^i \wedge \omega_\lambda = \int_{\partial \tilde{\Omega}} \delta_{\alpha\beta} p^\alpha a_i x_\epsilon^i n^\beta 3 d\sigma - \int_{\tilde{\Omega}} d(p^\lambda a_i) x_\epsilon^i \wedge \omega_\lambda,$$

we find $J'(0)$ as

$$\begin{aligned} & \int_{\tilde{\Omega}} \mathcal{L}_{x^j}(t, x(t), \hat{u}(t)) x_\epsilon^j(t, 0) \\ & + \int_{\tilde{\Omega}} \left(p^\lambda(t) \frac{\partial a_i}{\partial x^j}(x(t), \hat{u}(t)) dx^i(t) - d(p^\lambda a_j(x(t), \hat{u}(t))) \right) \wedge \omega_\lambda x_\epsilon^j(t, 0) \\ & + \int_{\partial \tilde{\Omega}} \delta_{\alpha\beta} p^\alpha a_i x_\epsilon^i(t, 0) n^\beta(t) d\sigma + \int_{\partial \tilde{\Omega}} g_{x^j}(x(t)) x_\epsilon^j(t, 0) d\sigma \\ & + \int_{\tilde{\Omega}} \left(\mathcal{L}_{u^a}(t, x(t), \hat{u}(t)) + p^\lambda(t) \frac{\partial a_i}{\partial u^a}(x(t), \hat{u}(t)) dx^i(t) \wedge \omega_\lambda \right) h^a(t). \end{aligned}$$

We select the costate function $p(t)$ as solution of the adjoint Pfaff equation (boundary value problem)

$$\begin{aligned} \mathcal{L}_{x^j}(t, x(t), \hat{u}(t)) + \left(p^\lambda(t) \frac{\partial a_i}{\partial x^j}(x(t), \hat{u}(t)) dx^i(t) - d(p^\lambda a_j(x(t), \hat{u}(t))) \right) \wedge \omega_\lambda &= 0, \\ \left(\delta_{\alpha\beta} p^\alpha(t) n^\beta(t) a_i(x(t), \hat{u}(t)) + g_{x^i}(x(t)) \right) |_{\partial \Omega} &= 0. \end{aligned}$$

On the other hand, we need $J'(0) = 0$ for all $h(t) = (h^a(t))$. Since the variation h is arbitrary, we get (critical point condition)

$$\frac{\partial \mathcal{L}}{\partial u^a}(t, x(t), \hat{u}(t)) + p^\lambda(t) \frac{\partial a_i}{\partial u^a}(x(t), \hat{u}(t)) dx^i(t) \wedge \omega_\lambda = 0.$$

The foregoing equations (14) and (15) can be written

$$\frac{\partial \mathcal{H}}{\partial x^j} - d(p^\lambda a_j) \wedge \omega_\lambda = 0, \quad \frac{\partial \mathcal{H}}{\partial u^a} = 0.$$

Let us start with a

1.4.2 Multitime optimal control problem of maximizing a curvilinear integral functional

Find

$$(16) \quad \max_{u(\cdot)} I(u(\cdot)) = \int_{\Gamma_{t_0 t_1}} L_\alpha(t, x(t), u(t)) dt^\alpha + g(x(t_1))$$

subject to

$$(17) \quad a_i(x(t), u(t)) dx^i(t) = 0, \text{ a.e. } t \in \Omega_{t_0 t_1}, \quad x(t_0) = x_0.$$

It is supposed that $L_\alpha : \Omega_{t_0 t_1} \times A \times U \rightarrow R$ and $a_i : A \times U \rightarrow R$, $i = 1, \dots, n$ are C^2 functions. Ingredients: $\mathcal{L} = L_\alpha(t, x(t), u(t))dt^\alpha$ is an 1-form, A is a bounded and closed subset of R^n , containing the images of the m -sheets $x(t)$, $t \in \Omega_{t_0 t_1}$ of the controlled system, and x_0 and x_1 are the initial and final states of the m -sheet $x(t)$ in the controlled system. The set, in which the control functions u^a takes their values, is called as U , which is a bounded and closed subset of R^k .

Let us find the first order necessary conditions for an optimal pair (x, u) . We fix the control $u(t)$ and variate the state $x(t)$ into $x(t, \epsilon)$, $\epsilon = (\epsilon^1, \dots, \epsilon^m)$. We find the *multitime infinitesimal deformation (Pfaff) system*

$$\frac{\partial a_i}{\partial x^j}(x, u)y_\alpha^j(t)dx^i + a_i(x, u)dy_\alpha^i = 0$$

of the nonholonomic constraint $a_i(x(t), u(t))dx^i(t) = 0$. It follows the *multitime adjoint Pfaff system*

$$d(pa_j(x, u)) = \frac{\partial(pa_i dx^i)}{\partial x^j},$$

whose solution $p(t)$ is called the *costate vector*. Here, the symbol d in the left hand member of the adjoint equation means the differentiation with respect to p and x .

We use the Lagrangian 1-form

$$\mathcal{L} = L_\alpha(t, x(t), u(t))dt^\alpha.$$

Introducing a *costate variable or Lagrange multiplier* p , we build a *Hamiltonian 1-form*

$$\mathcal{H} = \mathcal{L}(t, x(t), u(t)) + p(t)a_i(x(t), u(t))dx^i(t).$$

Theorem (Multitime maximum principle) *Suppose that the problem of maximizing the functional (16) constrained by (17) has an interior optimal solution $\hat{u}(t)$, which determines the optimal evolution $x(t)$. Then there exists a costate function $(p(t))$ such that*

$$(18) \quad \frac{\partial \mathcal{H}}{\partial p} = a_i(x(t), u(t))dx^i(t) = 0,$$

the function $(p(t))$ is the unique solution of the following Pfaff system (adjoint system)

$$(19) \quad d(pa_j(x, u)) = \frac{\partial \mathcal{H}}{\partial x^j}$$

and the critical point conditions

$$(20) \quad \mathcal{H}_{u^a}(t, x(t), u(t), p(t)) = 0, \quad a = \overline{1, k}$$

hold.

Proof We use the Hamiltonian 1-form \mathcal{H} . The solutions of the foregoing problem are between the solutions of the free maximization problem of the curvilinear integral functional

$$J(u(\cdot)) = \int_{\tilde{\Gamma}} \mathcal{H}(t, x(t), u(t)) + g(x(t_1)),$$

where $\tilde{\Gamma} = (\Gamma_{t_0 t_1}, x(\Gamma_{t_0 t_1})) \subset R_+^m \times R^n$.

Suppose that there exists a continuous control $\hat{u}(t)$ defined over the interval $\Omega_{t_0 t_1}$ with $\hat{u}(t) \in \text{Int } U$ which is an optimum point in the previous problem. We consider a control variation $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t)$, where h is an arbitrary continuous vector function, and a state variation $x(t, \epsilon)$, $t \in \Omega_{t_0 t_1}$, connected by

$$a_i(x(t, \epsilon), u(t, \epsilon)) dx^i(t, \epsilon) = 0, \quad x(t, 0) = x(t), \quad x(t_0, 0) = x_0.$$

Since $\hat{u}(t) \in \text{Int } \mathcal{U}$ and a continuous function over a compact set $\Omega_{t_0 t_1}$ is bounded, there exists $\epsilon_h > 0$ such that $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t) \in \text{Int } U$, $\forall |\epsilon| < \epsilon_h$. This ϵ is used in our variational arguments.

For $|\epsilon| < \epsilon_h$, we define the function

$$\begin{aligned} J(\epsilon) &= \int_{\tilde{\Gamma}(\epsilon)} \mathcal{H}(t, x(t, \epsilon), u(t, \epsilon), p(t)) + g(x(t_1, \epsilon)) \\ &= \int_{\tilde{\Gamma}(\epsilon)} \mathcal{L}(t, x(t, \epsilon), u(t, \epsilon)) + p(t) a_i(x(t, \epsilon), u(t, \epsilon)) dx^i(t, \epsilon) + g(x(t_1, \epsilon)). \end{aligned}$$

Differentiating with respect to ϵ , it follows

$$J'(\epsilon) = \int_{\tilde{\Gamma}(\epsilon)} \mathcal{L}_{x^j}(t, x(t, \epsilon), u(t, \epsilon)) x_\epsilon^j(t, \epsilon) + g_{x^j}(x(t_1, \epsilon)) x_\epsilon^j(t, \epsilon)$$

$$\begin{aligned}
& + \int_{\tilde{\Gamma}(\epsilon)} p(t) \frac{\partial a_i}{\partial x^j}(x(t, \epsilon), u(t, \epsilon)) x_\epsilon^j(t, \epsilon) dx^i(t, \epsilon) \\
& + \int_{\tilde{\Gamma}(\epsilon)} p(t) a_i(x(t, \epsilon), u(t, \epsilon)) dx_\epsilon^i(t, \epsilon) \\
& + \int_{\tilde{\Gamma}(\epsilon)} \mathcal{L}_{u^a}(t, x(t, \epsilon), u(t, \epsilon)) h^a(t) \\
& + \int_{\tilde{\Gamma}(\epsilon)} p(t) \frac{\partial a_i}{\partial u^a}(x(t, \epsilon), u(t, \epsilon)) h^a(t) dx^i(t, \epsilon).
\end{aligned}$$

Evaluating at $\epsilon = 0$, we find $\tilde{\Gamma}(0) = \tilde{\Gamma}$ and

$$\begin{aligned}
J'(0) &= \int_{\tilde{\Gamma}} \left(\mathcal{L}_{x^j}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial x^j}(x(t), \hat{u}(t)) dx^i(t) \right) x_\epsilon^j(t, 0) \\
&+ \int_{\tilde{\Gamma}} p(t) a_i(x(t), \hat{u}(t)) dx_\epsilon^i(t, 0) + g_{x^j}(x(t_1)) x_\epsilon^j(t, 0) \\
&+ \int_{\tilde{\Gamma}} \left(\mathcal{L}_{u^a}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial u^a}(x(t), \hat{u}(t)) dx^i(t) \right) h^a(t),
\end{aligned}$$

where $x(t)$ is the m -sheet of the state variable corresponding to the optimal control $\hat{u}(t)$.

To evaluate the curvilinear integral

$$\int_{\tilde{\Gamma}} p(t) a_i(x(t), \hat{u}(t)) dx_\epsilon^i(t, 0),$$

we integrate by parts, via the formula

$$d(pa_i x_\epsilon^i) = x_\epsilon^i d(pa_i) + pa_i dx_\epsilon^i,$$

obtaining

$$\begin{aligned}
\int_{\tilde{\Gamma}} pa_i dx_\epsilon^i &= \int_{\tilde{\Gamma}} d(pa_i x_\epsilon^i) - \int_{\tilde{\Gamma}} d(pa_i) x_\epsilon^i \\
&= (p(t) a_i(x(t), \hat{u}(t)) x_\epsilon^i(t, 0))|_{t_0}^{t_1} - \int_{\tilde{\Gamma}} d(p(t) a_i(x(t), \hat{u}(t))) x_\epsilon^i,
\end{aligned}$$

where the symbol d in the last integral means the differentiation with respect to p and x . We find

$$J'(0) = \int_{\tilde{\Gamma}} \mathcal{L}_{x^j}(t, x(t), \hat{u}(t)) x_\epsilon^j(t, 0)$$

$$\begin{aligned}
& + \int_{\tilde{\Gamma}} \left(p(t) \frac{\partial a_i}{\partial x^j}(x(t), \hat{u}(t)) dx^i(t) - d(p(t) a_j(x(t), \hat{u}(t))) \right) x_\epsilon^j(t, 0) \\
& \quad + (p(t) a_i(x(t), \hat{u}(t)) x_\epsilon^i(t, 0))|_{t_0}^{t_1} + g_{x^j}(x(t_1)) x_\epsilon^j(t, 0) \\
& + \int_{\tilde{\Gamma}} \left(\mathcal{L}_{u^a}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial u^a}(x(t), \hat{u}(t)) dx^i(t) \right) h^a(t).
\end{aligned}$$

We select the costate function $p(t)$ as solution of the adjoint Pfaff equation (terminal value problem)

$$\mathcal{L}_{x^j}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial x^j}(x(t), \hat{u}(t)) dx^i(t) - d(p(t) a_j(x(t), \hat{u}(t))) = 0,$$

subject to $p(t_1) a_i(x(t_1), \hat{u}(t_1)) + g_{x^i}(x(t_1)) = 0$. On the other hand, we need $J'(0) = 0$ for all $h(t) = (h^a(t))$. Since the variation h is arbitrary, we get (critical point condition)

$$\frac{\partial \mathcal{L}}{\partial u^a}(t, x(t), \hat{u}(t)) + p(t) \frac{\partial a_i}{\partial u^a}(x(t), \hat{u}(t)) dx^i(t) = 0.$$

The foregoing equations (19) and (20) can be written

$$\frac{\partial \mathcal{H}}{\partial x^j} = d(p a_j), \quad \frac{\partial \mathcal{H}}{\partial u^a} = 0.$$

2 Optimal control on distributions described by vector fields

2.1 Infinitesimal deformations and adjointness on distributions

The same distribution D can be described in terms of smooth vector fields (or generators),

$$(16) \quad D = \text{span}\{X_a(x) \mid a_i(x) X_a^i = 0, a = 1, \dots, n-1\},$$

if and only if $n \geq 3$. Any vector field Y in D can be written in the form $Y(x) = u^a(x) X_a(x)$.

Let $x(t)$ be a curve solution of the differential system

$$\dot{x}(t) = u^a(x(t)) X_a(x(t)).$$

Let $x(t; \epsilon)$ be a differentiable variation of $x(t)$, i.e.,

$$\dot{x}(t; \epsilon) = u^a(x(t; \epsilon))X_a(x(t; \epsilon)), \quad x(t; 0) = x(t).$$

Denoting $y^i(t) = \frac{\partial x^i}{\partial \epsilon}(t; 0)$, we find the *single-time infinitesimal deformation system*

$$(17) \quad \dot{y}^j(t) = \left(\frac{\partial u^a}{\partial x^i}(x(t))X_a^j(x(t)) + u^a(x(t))\frac{\partial X_a^j}{\partial x^i}(x(t)) \right) y^i(t).$$

The *single-time adjoint (dual) system* is

$$(18) \quad \dot{p}_k(t) = - \left(\frac{\partial u^a}{\partial x^k}(x(t))X_a^j(x(t)) + u^a(x(t))\frac{\partial X_a^j}{\partial x^k}(x(t)) \right) p_j(t),$$

whose solution $p = (p_k)$ is called the *costate vector*. The foregoing PDE systems (17) and (18) are *adjoint (dual)* in the sense of *constant interior product of solutions*, i.e., the scalar product $p_k y^k$ is a first integral.

Let $x(t)$ be an m -sheet integral submanifold of the distribution D , i.e., a solution of the multitime partial differential system

$$\frac{\partial x}{\partial t^\alpha}(t) = u_\alpha^a(x(t))X_a(x(t)), \quad \alpha = 1, \dots, m < n - 1.$$

Let $\epsilon = (\epsilon^\alpha)$, $\alpha = 1, \dots, m$ and let $x(t; \epsilon)$ be a differentiable variation of $x(t)$, i.e.,

$$\frac{\partial x}{\partial t^\alpha}(t; \epsilon) = u_\alpha^a(x(t; \epsilon))X_a(x(t; \epsilon)), \quad x(t; 0) = x(t).$$

Introducing the vector fields $y_\alpha^i(t) = \frac{\partial x^i}{\partial \epsilon^\alpha}(t; 0)$, we find the *multitime infinitesimal deformation system*

$$(19) \quad \frac{\partial y_\alpha^j}{\partial t^\beta}(t) = \left(\frac{\partial u_\beta^a}{\partial x^i}(x(t))X_a^j(x(t)) + u_\beta^a(x(t))\frac{\partial X_a^j}{\partial x^i}(x(t)) \right) y_\alpha^i(t).$$

The *multitime adjoint (dual) system* is

$$(20) \quad \frac{\partial p_k^\alpha}{\partial t^\beta}(t) = - \left(\frac{\partial u_\beta^a}{\partial x^k}(x(t))X_a^j(x(t)) + u_\beta^a(x(t))\frac{\partial X_a^j}{\partial x^k}(x(t)) \right) p_j^\alpha(t),$$

whose solution $p = (p_k^\alpha)$ is called the *costate matrix*. The foregoing PDE systems (19) and (20) are *adjoint (dual)* in the sense of *constant interior product of solutions*, i.e., the scalar product $p_k^\alpha y_\alpha^k$ is a first integral.

Of course, taking the trace, we can define the *costate matrix* $p : \Omega_{0T} \rightarrow R^{mn}$, $p = (p_k^\alpha)$, as the solution of the *divergence adjoint PDE system* (trace)

$$(21) \quad \frac{\partial p_k^\alpha}{\partial t^\alpha}(t) = - \left(\frac{\partial u_\alpha^a}{\partial x^k}(x(t)) X_a^j(x(t)) + u_\alpha^a(x(t)) \frac{\partial X_a^j}{\partial x^k}(x(t)) \right) p_j^\alpha(t).$$

But than, the PDEs systems (19) and (20) are *adjoint (dual)* in the sense of *zero total divergence* of the tensor field $Q_\beta^\alpha = p_k^\alpha y_\beta^k$ produced by their solutions. The divergence dual PDE system (21) has solutions since it contains n PDEs with nm unknown functions p_i^α . We can select a solution of the gradient form $p_k^\alpha(t) = \frac{\partial v^\alpha}{\partial x^k}(t, x(t))$.

Remark The *multitime adjoint Pfaff system* can be defined independent on the dimension of the parameter ϵ . Particularly, the *multitime adjoint Pfaff system* can be

$$(*) \quad \frac{\partial p_k}{\partial t^\beta}(t) = - \left(\frac{\partial u_\beta^a}{\partial x^k}(x(t)) X_a^j(x(t)) + u_\beta^a(x(t)) \frac{\partial X_a^j}{\partial x^k}(x(t)) \right) p_j(t).$$

2.2 Single-time optimal control problems on a distribution

Let

$$D = \text{span}\{X_a(x) \mid a_i(x) X_a^i = 0, a = 1, \dots, n-1\}.$$

be a distribution on R^n and $x(t)$, $t \in I = [t_0, t_1]$, be an integral curve of the driftless control system

$$dx(t) = u^a(x(t)) X_a(x(t)) dt.$$

A *single-time optimal control problem* is defined to be *maximizing the functional*

$$(22) \quad I(u(\cdot)) = \int_{t_0}^{t_1} L(t, x(t), u(x(t))) dt$$

subject to

$$(23) \quad dx(t) = u^a(x(t)) X_a(x(t)) dt, \text{ a.e. } t \in I = [t_0, t_1], x(t_0) = x_1, x(t_1) = x_1.$$

It is supposed that $L : I \times A \times U \rightarrow R$ is a C^2 function and $X_a : A \rightarrow R$, $a = 1, \dots, n-1$ are C^2 functions. Ingredients: A is a bounded and closed subset of R^n , which the trajectory of controlled system is constrained to stay for $t \in I$, and x_0 and x_1 are the initial and final states of the trajectory $x(t)$ in the controlled system. The set in which the control functions u^a takes their values in it, is called as U , which is a bounded and closed subset of R^{n-1} . The map u is assumed to be piecewise smooth or piecewise analytic. Such maps are called *admissible* and the space \mathcal{U} of all such maps is called the *set of admissible controls*.

Let us find the first order necessary conditions for an optimal pair (x, u) . Firstly, the *single-time infinitesimal deformation (Pfaff) equation* of the constraint $dx(t) = u^a(x(t))X_a(x(t))dt$ is the system (17) the *single-time adjoint Pfaff equation* is the system (18).

The control variables may be *open-loop* $u^a(t)$, depending directly on the time variable t , or *closed-loop* (or *feedback*) $u^a(x(t))$, depending on the state $x(t)$.

Open-loop control variables

To simplify, we accept an open-loop control $u^a(t)$. Using the Lagrangian 1-form

$$\mathcal{L}(t, x(t), u(t), p(t)) = L(t, x(t), u(t))dt + p_i(t)[u^a(t)X_a^i(x(t))dt - dx^i(t)],$$

we build the *Hamiltonian 1-form*

$$\mathcal{H} = L(t, x(t), u(t))dt + p_i(t)u^a(t)X_a^i(x(t))dt.$$

Theorem (Single-time maximum principle) *Suppose that the problem of maximizing the functional (22) constrained by (23) has an interior optimal solution $\hat{u}(t)$, which determines the optimal evolution $x(t)$. Then there exists a costate vector $p(t) = (p_i(t))$ such that*

$$(24) \quad dx^i = \frac{\partial \mathcal{H}}{\partial p_i},$$

the function $p(t)$ is the unique solution of the following Pfaff system (adjoint system)

$$(25) \quad dp_i = -\frac{\partial \mathcal{H}}{\partial x^i}$$

and the critical point conditions

$$(26) \quad \mathcal{H}_{u^a}(t, x(t), u(t), p(t)) = 0, \quad a = \overline{1, n-1}$$

hold.

Proof We use the Lagrangian 1-form \mathcal{L} . The solutions of the forgoing problem are between the solutions of the free maximization problem of the curvilinear integral functional

$$J(u(\cdot)) = \int_{\tilde{\Gamma}} \mathcal{L}(t, x(t), u(t), p(t)),$$

where $\tilde{\Gamma} = ([t_0, t_1], x([t_0, t_1])) \subset R_+ \times R^n$.

Suppose that there exists a continuous control $\hat{u}(t)$ defined over the interval I with $\hat{u}(t) \in \text{Int } U$ which is an optimum point in the previous problem. Now consider a variation $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t)$, where h is an arbitrary continuous vector function. Since $\hat{u}(t) \in \text{Int } U$ and a continuous function over a compact set I is bounded, there exists $\epsilon_h > 0$ such that $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t) \in \text{Int } U$, $\forall |\epsilon| < \epsilon_h$. This ϵ is used in our variational arguments.

Define $x(t, \epsilon)$ as the 1-sheet of the state variable corresponding to the control variable $u(t, \epsilon)$, i.e.,

$$dx^i(t; \epsilon) = u^a(x(t; \epsilon)) X_a^i(x(t; \epsilon)) dt, \quad x(t; 0) = x(t).$$

and $x(t_0, 0) = x_0$, $x(t_1, 0) = x_1$. For $|\epsilon| < \epsilon_h$, we define the function

$$J(\epsilon) = \int_{\tilde{\Gamma}(\epsilon)} \mathcal{L}(t, x(t, \epsilon), u(t, \epsilon), p(t))$$

$$\int_{\tilde{\Gamma}(\epsilon)} L(t, x(t, \epsilon), u(t, \epsilon)) dt + p_i(t) [u^a(t, \epsilon) X_a^i(x(t, \epsilon)) dt - dx^i(t, \epsilon)].$$

Differentiating with respect to ϵ , it follows

$$\begin{aligned} J'(\epsilon) &= \int_{\tilde{\Gamma}(\epsilon)} \left(L_{x^j}(t, x(t, \epsilon), u(t, \epsilon)) + p_i(t) u^a(t, \epsilon) X_{ax^j}^i \right) x_\epsilon^j(t, \epsilon) dt \\ &\quad - \int_{\tilde{\Gamma}(\epsilon)} p_i(t) dx_\epsilon^i(t, \epsilon) \\ &\quad + \int_{\tilde{\Gamma}(\epsilon)} \left(L_{u^a}(t, x(t, \epsilon), u(t, \epsilon)) + p_i(t) X_a^i(x(t, \epsilon)) \right) h^a(t) dt. \end{aligned}$$

Evaluating at $\epsilon = 0$, we find $\tilde{\Gamma}(0) = \tilde{\Gamma}$ and

$$\begin{aligned} J'(0) &= \int_{\tilde{\Gamma}} \left(L_{x^j}(t, x(t), \hat{u}(t)) + p_i(t) \hat{u}^a(t) X_{ax^j}^i \right) x_\epsilon^j(t, 0) dt \\ &\quad - \int_{\tilde{\Gamma}} p_i(t) dx_\epsilon^i(t, 0) \\ &\quad + \int_{\tilde{\Gamma}} \left(L_{u^a}(t, x(t), \hat{u}(t) + p_i(t) X_a^i(x(t))) \right) h^a(t) dt, \end{aligned}$$

where $x(t)$ is the curve of the state variable corresponding to the optimal control $\hat{u}(t)$. Since the integral from the middle can be written

$$\int_{\tilde{\Gamma}} p_i(t) dx_\epsilon^i(t, 0) = p_i(t) x_\epsilon^i|_{\partial\tilde{\Gamma}} - \int_{\tilde{\Gamma}} x_\epsilon^i(t, 0) dp_i(t),$$

we find $J'(0)$ as

$$\begin{aligned} J'(0) &= \int_{\tilde{\Gamma}} \left(\left(L_{x^j}(t, x(t), \hat{u}(t)) + p_i(t) \hat{u}^a(t) X_{ax^j}^i \right) dt + dp_j \right) x_\epsilon^j(t, 0) \\ &\quad - p_i(t) x_\epsilon^i(t, 0) \Big|_{t_0}^{t_1} \\ &\quad + \int_{\tilde{\Gamma}} \left(L_{u^a}(t, x(t), \hat{u}(t) + p_i(t) X_a^i(x(t))) \right) h^a(t) dt, \end{aligned}$$

We select the costate function $p(t)$ as solution of the adjoint Pfaff equation (boundary value problem)

$$\left(L_{x^j}(t, x(t), \hat{u}(t)) + p_i(t) \hat{u}^a(t) X_{ax^j}^i \right) dt + dp_j = 0, \quad p(t_0) = 0, p(t_1) = 0.$$

On the other hand, we need $J'(0) = 0$ for all $h(t) = (h^a(t))$. Since the variation h is arbitrary, we get (critical point condition)

$$L_{u^a}(t, x(t), \hat{u}(t) + p_i(t) X_a^i(x(t))) = 0.$$

Example Consider the ODE system $\dot{x}^1(t) = x^{2^2}(t), \dot{x}^2(t) = u(t)$ generated by the vector fields $X = x^{2^2} \frac{\partial}{\partial x^1}, Y = \frac{\partial}{\partial x^2}$. We compute the Lie brackets

$$[X, Y] = -2x^2 \frac{\partial}{\partial x^1}, [[X, Y], Y] = 2 \frac{\partial}{\partial x^1}.$$

The vector fields Y and $[[X, Y], Y]$ are linearly independent. On the other hand, the x^1 -coordinate is increasing since $x^{2^2} \geq 0$. Consequently, the system is not really controllable.

Closed-loop control variables

Now, we accept a closed-loop control $u^a(x(t))$. Using the Lagrangian 1-form

$$\mathcal{L} = L(t, x(t), u(x(t)))dt + p_i(t)[u^a(x(t))X_a^i(x(t))dt - dx^i(t)],$$

we build the *Hamiltonian 1-form*

$$\mathcal{H} = L(t, x(t), u(x(t)))dt + p_i(t)u^a(x(t))X_a^i(x(t))dt.$$

Theorem (Single-time maximum principle) *Suppose that the problem of maximizing the functional (22) constrained by (23) has an interior optimal solution $\hat{u}(x(t))$, which determines the optimal evolution $x(t)$. Then there exists a costate vector $p(t) = (p_i(t))$ such that*

$$dx^i = \frac{\partial \mathcal{H}}{\partial p_i},$$

the function $p(t)$ is the unique solution of the following Pfaff system (adjoint system)

$$dp_i = -\frac{\partial \mathcal{H}}{\partial x^i}$$

and the critical point conditions

$$\mathcal{H}_{u^a}(t, x(t), u(x(t)), p(t)) = 0, \quad a = \overline{1, n-1}$$

hold.

Proof The new functional is

$$J(x(\cdot), u(x(\cdot))) = \int_{\tilde{\Gamma}} \mathcal{L}(t, x(t), u(x(t)), p(t)).$$

A variation $x(t, \epsilon)$ induces a variation $u(x(t, \epsilon)) = \hat{u}(x(t)) + \epsilon h(t)$. Then

$$J(\epsilon) = \int_{\tilde{\Gamma}(\epsilon)} \mathcal{L}(t, x(t, \epsilon), u(x(t, \epsilon)), p(t)).$$

It follows

$$\begin{aligned} J'(\epsilon) &= \int_{\tilde{\Gamma}(\epsilon)} L_{x^j}(t, x(t, \epsilon), u(x(t, \epsilon)))x_\epsilon^j(t, \epsilon)dt \\ &+ \int_{\tilde{\Gamma}(\epsilon)} \left(p_i(t)u_{x^j}^a X_a^i + p_i(t)u^a(x(t, \epsilon))X_{ax^j}^i \right) x_\epsilon^j(t, \epsilon)dt \end{aligned}$$

$$\begin{aligned}
& - \int_{\tilde{\Gamma}(\epsilon)} p_i(t) dx_\epsilon^i(t, \epsilon) \\
& + \int_{\tilde{\Gamma}(\epsilon)} \left(L_{u^a}(t, x(t, \epsilon), u(x(t, \epsilon))) + p_i(t) X_a^i(x(t, \epsilon)) \right) h^a(t) dt.
\end{aligned}$$

Evaluating at $\epsilon = 0$, we find $\tilde{\Gamma}(0) = \tilde{\Gamma}$ and

$$\begin{aligned}
J'(0) &= \int_{\tilde{\Gamma}} L_{x^j}(t, x(t), \hat{u}(x(t))) x_\epsilon^j(t, 0) dt \\
&+ \int_{\tilde{\Gamma}} \left(p_i(t) \hat{u}_{x^j}^a X_a^i + p_i(t) u^a(x(t)) X_{ax^j}^i \right) x_\epsilon^j(t, 0) dt \\
&- \int_{\tilde{\Gamma}} p_i(t) dx_\epsilon^i(t, 0) \\
&+ \int_{\tilde{\Gamma}} \left(L_{u^a}(t, x(t), \hat{u}(x(t))) + p_i(t) X_a^i(x(t)) \right) h^a(t) dt.
\end{aligned}$$

Since the integral from the middle can be written

$$\int_{\tilde{\Gamma}} p_i(t) dx_\epsilon^i(t, 0) = p_i(t) x_\epsilon^i|_{\partial\tilde{\Gamma}} - \int_{\tilde{\Gamma}} x_\epsilon^i(t, 0) dp_i(t),$$

we find $J'(0)$ as

$$\begin{aligned}
J'(0) &= \int_{\tilde{\Gamma}} \left(\left(L_{x^j}(t, x(t), \hat{u}(x(t))) + p_i(t) \hat{u}_{x^j}^a X_a^i + p_i(t) \hat{u}^a(x(t)) X_{ax^j}^i \right) dt + dp_j \right) x_\epsilon^j(t, 0) \\
&- p_i(t) x_\epsilon^i(t, 0) \Big|_{t_0}^{t_1} \\
&+ \int_{\tilde{\Gamma}} \left(L_{u^a}(t, x(t), \hat{u}(x(t))) + p_i(t) X_a^i(x(t)) \right) h^a(t) dt,
\end{aligned}$$

We select the costate function $p(t)$ as solution of the adjoint Pfaff equation (boundary value problem)

$$\left(L_{x^j}(t, x(t), \hat{u}(x(t))) + p_i(t) \hat{u}^a(x(t)) X_{ax^j}^i \right) dt + dp_j = 0, \quad p(t_0) = 0, p(t_1) = 0.$$

On the other hand, we need $J'(0) = 0$ for all $h(t) = (h^a(t))$. Since the variation h is arbitrary, we get (critical point condition)

$$L_{u^a}(t, x(t), \hat{u}(x(t))) + p_i(t) X_a^i(x(t)) = 0.$$

2.3 Multitime optimal control problems on a distribution

Let

$$D = \text{span}\{X_a(x) \mid a_i(x)X_a^i = 0, a = 1, \dots, n-1\},$$

$n \geq 3$, be a distribution on R^n and $x(t)$, $t \in \Omega_{t_0 t_1}$, be an m -sheet of the driftless control system

$$dx(t) = u_\alpha^a(x(t))X_a(x(t))dt^\alpha.$$

Let us start with a

2.3.1 Multitime optimal control problem of maximizing a multiple integral functional

Find

$$(27) \quad \max_{u(\cdot)} I(u(\cdot)) = \int_{\Omega_{t_0 t_1}} L(t, x(t), u(x(t))) \omega$$

subject to

$$(28) \quad dx(t) = u_\alpha^a(x(t))X_a(x(t))dt^\alpha, \text{ a.e. } t \in \Omega_{t_0 t_1}, x(t_0) = x_0, x(t_1) = x_1.$$

It is supposed that $L : \Omega_{t_0 t_1} \times A \times U \rightarrow R$ is a C^2 function and $u^a : A \rightarrow R^{n-1}$, $a = 1, \dots, n-1$, $X_a^i : A \rightarrow R^{(n-1)n}$, $a = 1, \dots, n-1$, $i = 1, \dots, n$ are C^2 functions. Ingredients: $\omega = dt^1 \cdots dt^m$ is the volume element, A is a bounded and closed subset of R^n , which the m -sheet of controlled system is constrained to stay for $t \in \Omega_{t_0 t_1}$, and x_0 and x_1 are the initial and final states of the m -sheet $x(t)$ in the controlled system. The set in which the control functions $u^a(t)$ takes their values in it, is called as U , which is a bounded and closed subset of R^{n-1} .

Let us find the first order necessary conditions for an optimal pair (x, u) . Firstly, the *multitime infinitesimal deformation (Pfaff) system* of the constraint (28) is (19), and the *multitime adjoint Pfaff system* is (20).

The control variables may be *open-loop* $u^a(t)$, depending directly on the multitime variable t , or *closed-loop* (or *feedback*) $u^a(x(t))$, depending on the state $x(t)$.

To simplify, we accept an open-loop control. Introducing the $(m - 1)$ -forms

$$\omega_\lambda = \frac{\partial}{\partial t^\lambda} \rfloor \omega,$$

a costate variable matrix or Lagrange multiplier matrix $p = p_i^\alpha(t) \frac{\partial}{\partial t^\alpha} \otimes dx^i$ is identified to the $(m - 1)$ -forms $p_i = p_i^\lambda(t) \omega_\lambda$. We use the Lagrangian m -form $\mathcal{L}(t, x(t), u(t), p(t)) = L(t, x(t), u(t))\omega + p_i^\lambda(t)[u_\alpha^i(t)X_a^i(x(t))dt^\alpha - dx^i(t)] \wedge \omega_\lambda$ and the Hamiltonian m -form

$$\mathcal{H} = L(t, x(t), u(t))\omega + p_i^\lambda(t)u_\alpha^i(t)X_a^i(x(t))dt^\alpha \wedge \omega_\lambda.$$

Theorem (Multitime maximum principle) *Suppose that the problem of maximizing the functional (27) constrained by (28) has an interior optimal solution $\hat{u}(t)$, which determines the optimal evolution $x(t)$. Then there exists a costate matrix $(p(t))$ such that*

$$(29) \quad dx \wedge \omega_\lambda = \frac{\partial \mathcal{H}}{\partial p^\lambda},$$

the function $(p(t))$ is the unique solution of the following Pfaff system (adjoint system)

$$(30) \quad dp_j^\lambda \wedge \omega_\lambda = -\frac{\partial \mathcal{H}}{\partial x^j}, \quad \delta_{\alpha\beta} p_i^\alpha(t) n^\beta(t) = 0$$

and the critical point conditions

$$(31) \quad \mathcal{H}_{u^a}(t, x(t), u(t), p(t)) = 0, \quad a = \overline{1, n-1}$$

hold.

Proof We use the Lagrangian m -form \mathcal{L} . The solutions of the foregoing problem are between the solutions of the free maximization problem of the functional

$$J(u(\cdot)) = \int_{\tilde{\Omega}} \mathcal{L}(t, x(t), u(t), p(t)),$$

where $\tilde{\Omega} = (\Omega_{t_0 t_1}, x(\Omega_{t_0 t_1})) \subset R_+^m \times R^n$.

Suppose that there exists a continuous control $\hat{u}(t)$ defined over the interval $\Omega_{t_0 t_1}$ with $\hat{u}(t) \in \text{Int } U$ which is an optimum point in the previous

problem. Now consider a variation $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t)$, where h is an arbitrary continuous vector function. Since $\hat{u}(t) \in \text{Int } \mathcal{U}$ and a continuous function over a compact set $\Omega_{t_0 t_1}$ is bounded, there exists $\epsilon_h > 0$ such that $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t) \in \text{Int } U$, $\forall |\epsilon| < \epsilon_h$. This ϵ is used in our variational arguments.

Define $x(t, \epsilon)$ as the m -sheet of the state variable corresponding to the control variable $u(t, \epsilon)$, i.e.,

$$dx^i(t; \epsilon) = u_\alpha^a(x(t; \epsilon)) X_a^i(x(t; \epsilon)) dt^\alpha, \quad x(t; 0) = x(t).$$

and $x(t_0, 0) = x_0$, $x(t_1, 0) = x_1$. For $|\epsilon| < \epsilon_h$, we define the function

$$\begin{aligned} J(\epsilon) &= \int_{\tilde{\Omega}(\epsilon)} \mathcal{L}(t, x(t, \epsilon), u(t, \epsilon), p(t)) \\ &= \int_{\tilde{\Omega}(\epsilon)} L(t, x(t), u(t)) \omega + p_i^\lambda(t) [u_\alpha^a(t) X_a^i(x(t)) dt^\alpha - dx^i(t)] \wedge \omega_\lambda. \end{aligned}$$

Differentiating with respect to ϵ , it follows

$$\begin{aligned} J'(\epsilon) &= \int_{\tilde{\Omega}(\epsilon)} \left(L_{x^j}(t, x(t, \epsilon), u(t, \epsilon)) + p_i^\alpha(t) u_\alpha^a(t, \epsilon) X_{ax^j}^i(x(t, \epsilon)) \right) x_\epsilon^j(t, \epsilon) \omega \\ &\quad - \int_{\tilde{\Omega}(\epsilon)} p_i^\lambda(t) dx_\epsilon^i(t, \epsilon) \wedge \omega_\lambda \\ &\quad + \int_{\tilde{\Omega}(\epsilon)} \left(L_{u_\lambda^a}(t, x(t, \epsilon), u(t, \epsilon) + p_i^\lambda(t) X_a^i(x(t, \epsilon))) \right) h_\lambda^a(t) \omega. \end{aligned}$$

Evaluating at $\epsilon = 0$, we find

$$\begin{aligned} J'(0) &= \int_{\tilde{\Omega}(\epsilon)} \left(L_{x^j}(t, x(t), \hat{u}(t)) + p_i^\alpha(t) \hat{u}_\alpha^a(t) X_{ax^j}^i(x(t)) \right) x_\epsilon^j(t, 0) \omega \\ &\quad - \int_{\tilde{\Omega}(\epsilon)} p_i^\lambda(t) dx_\epsilon^i(t, 0) \wedge \omega_\lambda \\ &\quad + \int_{\tilde{\Omega}(\epsilon)} \left(L_{u_\lambda^a}(t, x(t), \hat{u}(t) + p_i^\lambda(t) X_a^i(x(t))) \right) h_\lambda^a(t) \omega. \end{aligned}$$

where $x(t)$ is the m -sheet of the state variable corresponding to the optimal control $\hat{u}(t)$.

To evaluate the multiple integral

$$\int_{\tilde{\Omega}(\epsilon)} p_i^\lambda(t) dx_\epsilon^i(t, 0) \wedge \omega_\lambda,$$

we integrate by parts, via the formula

$$d(p_i^\lambda x_\epsilon^i \omega_\lambda) = (x_\epsilon^i dp_i^\lambda + p_i^\lambda dx_\epsilon^i) \wedge \omega_\lambda,$$

obtaining

$$\int_{\tilde{\Omega}(\epsilon)} p_i^\lambda dx_\epsilon^i \wedge \omega_\lambda = \int_{\tilde{\Omega}(\epsilon)} d(p_i^\lambda x_\epsilon^i \omega_\lambda) - \int_{\tilde{\Omega}(\epsilon)} x_\epsilon^i dp_i^\lambda \wedge \omega_\lambda.$$

Now we apply the Stokes integral formula

$$\int_{\tilde{\Omega}(\epsilon)} d(p_i^\lambda x_\epsilon^i \omega_\lambda) = \int_{\partial \tilde{\Omega}(\epsilon)} \delta_{\alpha\beta} p_i^\alpha x_\epsilon^i n^\beta d\sigma,$$

where $(n^\beta(t))$ is the unit normal vector to the boundary $\partial \tilde{\Omega}$. Since the integral from the middle can be written

$$\int_{\tilde{\Omega}(\epsilon)} p_i^\lambda dx_\epsilon^i \wedge \omega_\lambda = \int_{\partial \tilde{\Omega}(\epsilon)} \delta_{\alpha\beta} p_i^\alpha x_\epsilon^i n^\beta d\sigma - \int_{\tilde{\Omega}} x_\epsilon^i dp_i^\lambda \wedge \omega_\lambda,$$

we find $J'(0)$ as

$$\begin{aligned} J'(0) &= \int_{\tilde{\Omega}} \left(L_{x^j}(t, x(t), \hat{u}(t)) + p_i^\alpha(t) \hat{u}_\alpha^a(t) X_{ax^j}^i(x(t)) \right) x_\epsilon^j(t, 0) \omega \\ &\quad - \int_{\partial \tilde{\Omega}} \delta_{\alpha\beta} p_i^\alpha x_\epsilon^i(t, 0) n^\beta d\sigma + \int_{\tilde{\Omega}} dp_i^\lambda \wedge \omega_\lambda x_\epsilon^i(t, 0) \\ &\quad + \int_{\tilde{\Omega}} \left(L_{u_\lambda^a}(t, x(t), \hat{u}(t) + p_i^\lambda(t) X_a^i(x(t))) \right) h_\lambda^a(t) \omega. \end{aligned}$$

We select the costate function $p(t)$ as solution of the adjoint Pfaff equation (boundary value problem)

$$\left(L_{x^j}(t, x(t), \hat{u}(t)) + p_i^\alpha(t) \hat{u}_\alpha^a(t) X_{ax^j}^i(x(t)) \right) \omega + dp_j^\lambda \wedge \omega_\lambda = 0, \delta_{\alpha\beta} p_i^\alpha(t) n^\beta(t)|_{\partial \Omega} = 0.$$

On the other hand, we need $J'(0) = 0$ for all $h(t) = (h^a(t))$. Since the variation h is arbitrary, we get (critical point condition)

$$L_{u_\lambda^a}(t, x(t), \hat{u}(t) + p_i^\lambda(t) X_a^i(x(t))) = 0.$$

Let us start with a

2.3.2 Multitime optimal control problem of maximizing a curvilinear integral functional

Find

$$(32) \quad \max_{u(\cdot)} I(u(\cdot)) = \int_{\Gamma_{t_0 t_1}} L_\alpha(t, x(t), u(x(t))) dt^\alpha$$

subject to

$$(33) \quad dx(t) = u_\alpha^a(x(t)) X_a(x(t)) dt^\alpha \text{ a.e. } t \in \Omega_{t_0 t_1}, \quad x(t_0) = x_0, \quad x(t_1) = x_1.$$

It is supposed that $L_\alpha : \Omega_{t_0 t_1} \times A \times U \rightarrow R$ and $u^a : A \rightarrow R^{n-1}$, $a = 1, \dots, n-1$, $X_a^i : A \rightarrow R^{(n-1)n}$, $a = 1, \dots, n-1$, $i = 1, \dots, n$ are C^2 functions. Ingredients: $\mathcal{L} = L_\alpha(t, x(t), u(t)) dt^\alpha$ is an 1-form, A is a bounded and closed subset of R^n , which the m -sheet of controlled system is constrained to stay for $t \in \Omega_{t_0 t_1}$, and x_0 and x_1 are the initial and final states of the m -sheet $x(t)$ in the controlled system. The set in which the control functions u^a takes their values in it, is called as U , which is a bounded and closed subset of R^{n-1} .

Let us find the first order necessary conditions for an optimal pair (x, u) . Firstly, the *multitime infinitesimal deformation (Pfaff) system* of the constraint (33) is (19), and the *multitime adjoint Pfaff system* is (*).

The control variables may be *open-loop* $u^a(t)$, depending directly on the multitime variable t , or *closed loop* (or *feedback*) $u^a(x(t))$, depending on the state $x(t)$.

To simplify, we accept an open loop control. Introducing a *costate variable vector or Lagrange multiplier* $p = (p_i(t))$, we build a Lagrangian 1-form

$$\mathcal{L}(t, x(t), u(t), p(t)) = L_\alpha(t, x(t), u(t)) dt^\alpha + p_i(t) [u_\alpha^a(t) X_a^i(x(t)) dt^\alpha - dx^i(t)].$$

and a *Hamiltonian 1-form*

$$\mathcal{H} = L_\alpha(t, x(t), u(t)) dt^\alpha + p_i(t) u_\alpha^a(t) X_a^i(x(t)) dt^\alpha.$$

Theorem (Multitime maximum principle) *Suppose that the problem of maximizing the functional (32) constrained by (33) has an interior optimal solution $\hat{u}(t)$, which determines the optimal evolution $x(t)$. Then there exists a costate function $(p(t))$ such that*

$$(34) \quad dx^i = \frac{\partial \mathcal{H}}{\partial p_i},$$

the function $(p(t))$ is the unique solution of the following Pfaff system (adjoint system)

$$(35) \quad dp_i = -\frac{\partial \mathcal{H}}{\partial x^i}$$

and the critical point conditions

$$(36) \quad \mathcal{H}_{u_\alpha^a}(t, x(t), u(t), p(t)) = 0, \quad a = \overline{1, n-1}, \alpha = 1, \dots, m$$

hold.

Proof We use the Lagrangian 1-form \mathcal{L} . The solutions of the foregoing problem are between the solutions of the free maximization problem of the curvilinear integral functional

$$J(u(\cdot)) = \int_{\tilde{\Gamma}} \mathcal{L}(t, x(t), u(t), p(t)),$$

where $\tilde{\Gamma} = (\Gamma_{t_0 t_1}, x(\Gamma_{t_0 t_1})) \subset R_+^m \times R^n$.

Suppose that there exists a continuous control $\hat{u}(t)$ defined over the interval $\Omega_{t_0 t_1}$ with $\hat{u}(t) \in \text{Int } U$ which is an optimum point in the previous problem. Now consider a variation $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t)$, where h is an arbitrary continuous vector function. Since $\hat{u}(t) \in \text{Int } \mathcal{U}$ and a continuous function over a compact set $\Omega_{t_0 t_1}$ is bounded, there exists $\epsilon_h > 0$ such that $u(t, \epsilon) = \hat{u}(t) + \epsilon h(t) \in \text{Int } U$, $\forall |\epsilon| < \epsilon_h$. This ϵ is used in our variational arguments.

Define $x(t, \epsilon)$ as the m -sheet of the state variable corresponding to the control variable $u(t, \epsilon)$, i.e.,

$$dx^i(t; \epsilon) = u_\alpha^a(x(t; \epsilon)) X_a^i(x(t; \epsilon)) dt^\alpha, \quad x(t; 0) = x(t)$$

and $x(t_0, 0) = x_0$, $x(t_1, 0) = x_1$. For $|\epsilon| < \epsilon_h$, we define the function

$$\begin{aligned} J(\epsilon) &= \int_{\tilde{\Gamma}(\epsilon)} \mathcal{L}(t, x(t, \epsilon), u(t, \epsilon), p(t)) \\ &= \int_{\tilde{\Gamma}(\epsilon)} L_\alpha(t, x(t, \epsilon), u(t, \epsilon)) dt^\alpha + p_i(t) [u_\alpha^a(t, \epsilon) X_a^i(x(t, \epsilon)) dt^\alpha - dx^i(t, \epsilon)]. \end{aligned}$$

Differentiating with respect to ϵ , it follows

$$J'(\epsilon) = \int_{\tilde{\Gamma}(\epsilon)} \left(L_{\alpha x^j}(t, x(t, \epsilon), u(t, \epsilon)) dt^\alpha + p_i(t) u_\alpha^a(t, \epsilon) X_{ax^j}^i(x(t, \epsilon)) dt^\alpha \right) x_\epsilon^j(t, \epsilon)$$

$$\begin{aligned}
& - \int_{\tilde{\Gamma}(\epsilon)} p_i(t) dx_\epsilon^i(t, \epsilon) \\
& + \int_{\tilde{\Gamma}(\epsilon)} \left(L_{\beta u_\alpha^a}(t, x(t, \epsilon), u(t, \epsilon)) dt^\beta + p_i(t) X_a^i(x(t, \epsilon)) dt^\alpha \right) h_\alpha^a(t).
\end{aligned}$$

Evaluating at $\epsilon = 0$, we find $\tilde{\Gamma}(0) = \tilde{\Gamma}$ and

$$\begin{aligned}
J'(0) &= \int_{\tilde{\Gamma}} \left(L_{\alpha x^j}(t, x(t), \hat{u}(t)) dt^\alpha + p_i(t) \hat{u}_\alpha^a(t) X_{ax^j}^i(x(t)) dt^\alpha \right) x_\epsilon^j(t, 0) \\
& - \int_{\tilde{\Gamma}} p_i(t) dx_\epsilon^i(t, 0) \\
& + \int_{\tilde{\Gamma}} \left(L_{\beta u_\alpha^a}(t, x(t), \hat{u}(t)) dt^\beta + p_i(t) X_a^i(x(t)) dt^\alpha \right) h_\alpha^a(t).
\end{aligned}$$

where $x(t)$ is the m -sheet of the state variable corresponding to the optimal control $\hat{u}(t)$.

To evaluate the curvilinear integral

$$\int_{\tilde{\Gamma}} p_i(t) dx_\epsilon^i(t, 0),$$

we integrate by parts, via the formula

$$d(p_i x_\epsilon^i) = x_\epsilon^i dp_i + p_i dx_\epsilon^i,$$

obtaining

$$\int_{\tilde{\Gamma}} p_i dx_\epsilon^i = (p_i(t) x_\epsilon^i(t, 0))|_{t_0}^{t_1} - \int_{\tilde{\Gamma}} (dp_i) x_\epsilon^i.$$

We find

$$\begin{aligned}
J'(0) &= \int_{\tilde{\Gamma}} \left(L_{\alpha x^j}(t, x(t), \hat{u}(t)) dt^\alpha + p_i(t) \hat{u}_\alpha^a(t) X_{ax^j}^i(x(t)) dt^\alpha + dp_j \right) x_\epsilon^j(t, 0) \\
& - (p_i(t) x_\epsilon^i(t, 0))|_{t_0}^{t_1} \\
& + \int_{\tilde{\Gamma}} \left(L_{\beta u_\alpha^a}(t, x(t), \hat{u}(t)) dt^\beta + p_i(t) X_a^i(x(t)) dt^\alpha \right) h_\alpha^a(t).
\end{aligned}$$

We select the costate function $p(t)$ as solution of the adjoint Pfaff equation (boundary value problem)

$$L_{\alpha x^j}(t, x(t), \hat{u}(t)) dt^\alpha + p_i(t) \hat{u}_\alpha^a(t) X_{ax^j}^i(x(t)) dt^\alpha + dp_j = 0.$$

On the other hand, we need $J'(0) = 0$ for all $h(t) = (h_\alpha^a(t))$. Since the variation h is arbitrary, we get (critical point condition)

$$L_{\beta u_\alpha^a}(t, x(t), \hat{u}(t))dt^\beta + p_i(t)X_a^i(x(t))dt^\alpha = 0.$$

Example: Nonholonomic control of torsion of a cylinder or prism

Suppose the torsion of a cylinder or prism is described by the controlled Pfaff equation

$$dz = (y + u(x, y))dx + (-x + v(x, y))dy, \quad z(0, 0) = 0, z(x_0, y_0) = z_0,$$

where the control (u, v) is not subject to constraints. If the complete integrability condition $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2$ is verified identically, then we have a holonomic evolution. Otherwise, we have a nonholonomic evolution. Using the controlled Pfaff equation as constraint, we want to minimize the functional

$$\int_\Gamma (z(x, y) + u(x, y)^2)c_1 dx + (z(x, y) + v(x, y)^2)c_2 dy,$$

where Γ is a C^1 curve joining the points $(0, 0)$ and (x_0, y_0) , and c_1, c_2 are constants. The minimization of the previous integral is equivalent to the maximization of the cost functional

$$P(u(\cdot), v(\cdot)) = - \int_\Gamma (z(x, y) + u(x, y)^2)c_1 dx + (z(x, y) + v(x, y)^2)c_2 dy$$

subject to controlled Pfaff equation.

Let us find the optimal manifold (surface or curve) of evolution, using the two-variable maximum principle theory. For that we introduce the 1-forms:

$$\begin{aligned} \omega &= (y + u)dx + (-x + v)dy - dz \text{ (evolution 1-form)} \\ \omega^0 &= -(z + u^2)c_1 dx - (z + v^2)c_2 dy \text{ (running cost 1-form)} \\ \eta &= p_0 \omega^0 + p \omega \text{ (control 1-form)}. \end{aligned}$$

Taking $p_0 = 1$, we obtain

$$\eta = -(z + u^2)c_1 dx - (z + v^2)c_2 dy + p((y + u)dx + (-x + v)dy - dz).$$

The adjoint equation $dp = -\frac{\partial}{\partial z}\eta = c_1 dx + c_2 dy$, $p(0, 0) = 0$ has the solution $p(x, y) = c_1 x + c_2 y$. The maximization condition

$$H_1(x, y, z(x, y), p(x, y), u(x, y)) = \max_{u, v} \{-(z(x, y) + u^2)c_1 + p(x, y)(y + u)\}$$

$$H_2(x, y, z(x, y), p(x, y), v(x, y)) = \max_{u, v} \{-(z(x, y) + v^2)c_1 + p(x, y)(y + u)\}$$

gives the optimal law

$$u(x, y) = \frac{p(x, y)}{2c_1}, \quad v(x, y) = \frac{p(x, y)}{2c_2}.$$

Replacing in the evolution Pfaff equation we obtain

$$dz = \left(y + \frac{c_1x + c_2y}{2c_1}\right) dx + \left(-x + \frac{c_1x + c_2y}{2c_2}\right) dy.$$

1) If the complete integrability condition

$$\frac{\partial}{\partial y} \left(y + \frac{c_1x + c_2y}{2c_1}\right) = \frac{\partial}{\partial x} \left(-x + \frac{c_1x + c_2y}{2c_2}\right),$$

i.e., $4c_1c_2 + c_2^2 - c_1^2 = 0$ is satisfied, then the evolution surface is

$$z = xy + \frac{x^2}{4} + \frac{y^2}{4} + \frac{c_2}{2c_1}xy.$$

2) If $4c_1c_2 + c_2^2 - c_1^2 \neq 0$, then the Pfaff evolution equation admits only solutions which are curves (nonholonomic surface in R^3): $x = x(t), y = y(t), z = z(t), t \in I$, with

$$x = x(t), y = y(t) \text{ (given arbitrary)}$$

$$\frac{dz}{dt} = y(t)x'(t) - x(t)y'(t) + (c_1x(t) + c_2y(t)) \left(\frac{x'(t)}{2c_1} + \frac{y'(t)}{2c_2} \right).$$

In this case, for determining $z = z(t)$, we must take $x = x(t), y = y(t), t \in I$ as a parametrization of the curve Γ from the cost functional. In fact, the problem is reduced to optimization of a simple integral constrained by a differential equation.

3 Curvilinear integral functionals depending on curves

Let $\Omega_{t_0t_1} \subset R^m$, let $\Gamma_{t_0t_1} \subset \Omega_{t_0t_1}$ be a C^1 curve and

$$J(\Gamma_{t_0t_1}; x(\cdot)) = \int_{\Gamma_{t_0t_1}} L_\alpha(t, x(t), x_\gamma(t)) dt^\alpha$$

be a curvilinear integral functional depending on the curve $\Gamma_{t_0 t_1}$. Consider a variation $\Gamma_{t_0 t_1}(\epsilon) : t = (t^\alpha(\tau, \epsilon))$ of the curve $\Gamma_{t_0 t_1} : t = (t^\alpha(\tau))$, with the same endpoints. Suppose $L_\alpha(t, x(t), x_\gamma(t))dt^\alpha$ is stationary with respect to ϵ . Then

$$J(\epsilon) = \int_{\Gamma_{t_0 t_1}(\epsilon)} L_\alpha(t, x(t), x_\gamma(t))dt^\alpha.$$

The closed curve $C = \Gamma_{t_0 t_1}(\epsilon) \cup \Gamma_{t_1 t_0}(0)$ is the boundary of a surface S . We evaluate $J(\epsilon) - J(0)$, using Stokes formula,

$$\begin{aligned} J(\epsilon) - J(0) &= \int_{\Gamma_{t_0 t_1}(\epsilon)} L_\alpha(t, x(t), x_\gamma(t))dt^\alpha - \int_{\Gamma_{t_0 t_1}(0)} L_\alpha(t, x(t), x_\gamma(t))dt^\alpha \\ &= \int_{\Gamma_{t_0 t_1}(\epsilon)} L_\alpha(t, x(t), x_\gamma(t))dt^\alpha + \int_{\Gamma_{t_1 t_0}(0)} L_\alpha(t, x(t), x_\gamma(t))dt^\alpha \\ &= \int_C L_\alpha(t, x(t), x_\gamma(t))dt^\alpha = \int_S d(L_\alpha(t, x(t), x_\gamma(t))dt^\alpha) = \int_S D_\beta L_\alpha dt^\beta \wedge dt^\alpha \\ &= \frac{1}{2} \int_S (D_\alpha L_\beta - D_\beta L_\alpha) dt^\alpha \wedge dt^\beta. \end{aligned}$$

Now we use the variation vector field $\frac{\partial t^\alpha}{\partial \epsilon}(\tau, \epsilon)|_{\epsilon=0} = \xi^\alpha(\tau)$. Replacing $dt^\alpha = \epsilon \xi^\alpha$, the surface integral is transformed to a curvilinear integral

$$= \frac{\epsilon}{2} \int_{\Gamma_{t_0 t_1}(0)} (D_\alpha L_\beta - D_\beta L_\alpha)(\xi^\alpha dt^\beta - \xi^\beta dt^\alpha) = \epsilon \int_{\Gamma_{t_0 t_1}(0)} (D_\alpha L_\beta - D_\beta L_\alpha) \xi^\alpha dt^\beta.$$

It follows

$$J'(0) = \int_{\Gamma_{t_0 t_1}(0)} (D_\alpha L_\beta - D_\beta L_\alpha) \xi^\alpha dt^\beta.$$

Suppose $\Gamma_{t_0 t_1}$ is a critical point of the functional, hence $J'(0) = 0, \forall \xi$. Consequently

$$(D_\alpha L_\beta - D_\beta L_\alpha)(t(\tau)) \frac{dt^\beta}{d\tau}(\tau) = 0.$$

If the curvilinear integral is path independent, then this relation is identically satisfied. If the curvilinear integral is path dependent, then the discussion depends on m since $a_{\alpha\beta} = D_\alpha L_\beta - D_\beta L_\alpha$ is an anti-symmetric matrix, and consequently its determinant $d = \det(a_{\alpha\beta})$ is either 0, for m odd, or ≥ 0 , for m even. For m -odd we have solutions, i.e., critical curves; for m -even, we have either no solution for $d > 0$ or solutions for $d = 0$. Since the differential system is of order one, the curve solution is determined only

by a single condition (the general bilocal problems have no solution). The extremum problems have sense only if we add supplementary conditions (an initial condition + an isoperimetric condition).

Variation Let $\Omega_{t_0 t_1} \subset R^m$, let $\Gamma_{t_0 t_1} \subset \Omega_{t_0 t_1}$ be a C^1 curve and

$$J(\Gamma_{t_0 t_1}; x(\cdot)) = \int_{\Gamma_{t_0 t_1}} L_\alpha(t, x(t), x_\gamma(t)) dt^\alpha$$

be a curvilinear integral functional depending on the curve $\Gamma_{t_0 t_1}$. Consider a variation $\Gamma_{t_0 t_1}(\epsilon) : t = (t^\alpha(\tau, \epsilon))$ of the curve $\Gamma_{t_0 t_1} : t = (t^\alpha(\tau))$, with the same endpoints. Denote $M_\alpha(t) = L_\alpha(t, x(t), x_\gamma(t))$.

Then

$$J(\epsilon) = \int_{\Gamma_{t_0 t_1}(\epsilon)} M_\alpha(t(\tau, \epsilon)) dt^\alpha(\tau, \epsilon).$$

To compute $J'(0)$, we use the variation vector field $\frac{\partial t^\alpha}{\partial \epsilon}(\tau, \epsilon)|_{\epsilon=0} = \xi^\alpha(\tau)$. From

$$J'(\epsilon) = \int_{\Gamma_{t_0 t_1}(\epsilon)} \frac{\partial M_\alpha}{\partial t^\beta}(t(\tau, \epsilon)) \frac{\partial t^\beta}{\partial \epsilon}(\tau, \epsilon) dt^\alpha(\tau, \epsilon) + M_\alpha(t(\tau, \epsilon)) d \frac{\partial t^\alpha}{\partial \epsilon}(\tau, \epsilon),$$

we obtain

$$J'(0) = \int_{\Gamma_{t_0 t_1}(0)} \frac{\partial M_\alpha}{\partial t^\beta}(t(\tau)) \xi^\beta(\tau) dt^\alpha(\tau) + M_\beta(t(\tau)) d\xi^\beta(\tau).$$

Integrating by parts, we find

$$J'(0) = M_\beta(t(\tau)) \xi^\beta(\tau)|_{\tau_0}^{\tau_1} + \int_{\Gamma_{t_0 t_1}(0)} \left(\frac{\partial M_\alpha}{\partial t^\beta} - \frac{\partial M_\beta}{\partial t^\alpha} \right) (t(\tau)) \xi^\beta(\tau) dt^\alpha(\tau).$$

Remark The variation of the function $x(t)$ has nothing to do with the variation of the curve.

4 Optimization of mechanical work on Riemannian manifolds

Let (M, g) be a Riemannian manifold and X a C^2 vector field on M . Let $x = (x^1, \dots, x^n)$ denote the local coordinates relative to a fixed local map (V, h) . Since $h : V \rightarrow R^n$ is a diffeomorphism, we denote by $\Omega_{x_0 x_1}$ a subset of

V diffeomorphic through h with the hyper-parallelepiped in R^n having $h(x_0)$ and $h(x_1)$ as diagonal points.

Let $\Gamma_{x_0x_1} : x^i = x^i(t), t \in [t_0, t_1]$ be an arbitrary C^1 curve on M which joins the points $x(t_0) = x_0, x(t_1) = x_1$. The functional

$$J(\Gamma_{x_0x_1}) = \int_{\Gamma_{x_0x_1}} g_{ij}(x) X^i(x) dx^j$$

is generated by the *mechanical work* produced by the force $\omega_j = g_{ij}(x) X^i(x)$ along the curve $\Gamma_{x_0x_1}$.

Let X be a nowhere zero vector field. X is called a *geodesic vector field* iff $\nabla_X X = 0$. Thus X is geodesic iff each of its integral curves is a geodesic.

Theorem *If X is a unit geodesic vector field and $\gamma_{x_0x_1}$ is a field line, then the curve $\gamma_{x_0x_1}$ is a maximum point of the functional $J(\Gamma_{x_0x_1})$ and the maximum value is the length of $\gamma_{x_0x_1}$.*

Proof Let us find

$$\max_{\Gamma_{x_0x_1}} J(\Gamma_{x_0x_1}) = \int_{\Gamma_{x_0x_1}} g_{ij}(x) X^i(x) dx^j,$$

where

$$g_{ij}(x) X^i(x) X^j(x) = 1.$$

The critical point condition, with respect to the curve $\Gamma_{x_0x_1}$, is

$$\left(g_{ij} \nabla_k X^i - g_{ik} \nabla_j X^i \right) (x(t)) \frac{dx^j}{dt}(t) = 0.$$

It is identically satisfied, because $\gamma_{x_0x_1}$ is a field line, the geodesic condition implies $\nabla_X X^i = 0$ and the condition of unit vector field gives $g_{ij} X^j \nabla_k X^i = 0$.

On the other hand, the inequality

$$|g_{ij}(x) X^i(x) dx^j| \leq \|X\| ds, \quad ds = \|dx\| = \sqrt{g_{ij} dx^i dx^j}$$

becomes an equality if $dx^j = X^j(x(t)) dt$, i.e., $x(t)$ is a field line of $X(x)$. Under the condition $\|X\| = 1$, the maximum value of the foregoing functional is the length of $\gamma_{x_0x_1}$.

5 Bang-bang control on distributions

The same distribution D can be described in terms of vector fields,

$$(22) \quad D = \text{span}\{X_a(x) \mid a_i(x)X_a^i = 0, a = 1, \dots, n-1\}.$$

Bang-bang control is an optimal or suboptimal piecewise constant control whose values are defined by bounds imposed on the amplitude of control components. The control changes its values according to the switching function which may be found using the maximum principle. The discontinuity of the bang-bang control leads to discontinuity of a value function for the considered optimal control problem. Typical problems with bang-bang optimal control include time and terminal cost optimal control for linear control systems. Bang-bang optimization offers a direct explanation for an otherwise perplexing observation and indicates that evolution is operating according to principles that every engineer knows. The black hole applications covered in this Section refer to the controllability of the ODE or PDE system by bang-bang controls.

5.1 Single-time bang-bang optimal control

Let $x(t)$, $t \in I = [0, \tau] \subset R$, be an integral curve of the distribution D . Any curve in the distribution $\Delta = \text{span}\{X_a, a = 1, \dots, n-1\}$ is a solution of the controlled ODE system

$$\dot{x}(t) = u^a(t)X_a(x(t)), \quad u(t) = (u^a(t)), \quad t \in [0, \tau], \quad (ODE)$$

called *driftless control system*.

(1) Time minimum problem Let $U = [-1, 1]^{n-1} \subset R^{n-1}$ be the control set. Given the starting point $x_0 \in R^n$, find an optimal control $u^*(\cdot)$ such that

$$I(u^*(\cdot)) = \min_{u(\cdot)} \int_0^\tau dt,$$

using (ODE) evolution as constraint. Since $\tau^* = I(u^*(\cdot))$, the optimal point τ^* ensures the minimum time to steer to the origin. This time optimum problem is equivalent to a controllability one.

Solution To prove the existence of a bang-bang control, we use the single-time Pontryagin Maximum Principle. The Hamiltonian $H(x, p, u) = -1 +$

$p_i X_a^i(x) u^a$ gives the adjoint ODE system $\dot{p}_j(t) = -p_i(t) \frac{\partial X_a^i}{\partial x^j}(x(t)) u^a(t)$. The extremum of the linear function $u \rightarrow H$ exists since each control variable belong to the interval $[-1, 1]$; for optimum, the control must be at a vertex of ∂U (see, linear optimization, simplex method). If $Q_a(t) = p_i(t) X_a^i(x(t))$, then the optimal control u^{*a} must be the function (bang-bang control)

$$u^{*a} = -\text{sign}(Q_a(t)) = \begin{cases} 1 & \text{for } Q_a(t) < 0 \\ \text{undetermined} & \text{for } Q_a(t) = 0 \text{ (singular control)} \\ -1 & \text{for } Q_a(t) > 0. \end{cases}$$

Suppose the Lebesgue measure of each set $\{t \in [0, \tau] : Q_a(t) = 0\}$ vanishes. Then the singular control is ruled out and the remaining possibilities are bang-bang controls. This optimal control is discontinuous since each component jumps from a minimum to a maximum and vice versa in response to each change in the sign of each $Q_a(t)$. The functions $Q_a(t)$ are called *switching functions*.

(2) Optimal terminal value Let $U = [-1, 1]^{n-1} \subset R^{n-1}$ be the control set. Suppose we have to

Minimize the terminal cost functional

$$Q(u(\cdot)) = x^n(t_1)$$

subject to the driftless control system

$$\dot{x}(t) = u^a(t) X_a(x(t)), \quad u(t) \in \mathcal{U}, \quad t \in [t_0, t_1]; \quad x(t_0) = x_0.$$

Solution Since the control Hamiltonian $H(x, p, u) = p_i X_a^i(x) u^a$ is linear in the control, the optimal control is a bang-bang. Automatically we find the optimal costate function and the optimal evolution.

5.2 Multitime bang-bang optimal control

Let $\Omega_{0\tau}$ be the hyperparallelepiped determined by two opposite diagonal points $0 = (0, \dots, 0)$ and $\tau = (\tau^1, \dots, \tau^m)$ in R_+^m , endowed with the product order. Let $x(t)$, $t \in \Omega_{0\tau} \subset R_+^m$, be an integral m -sheet of the distribution D , i.e., a solution of a multitime piecewise completely integrable PDE system

$$\frac{\partial x}{\partial t^\alpha}(t) = u_\alpha^a(t) X_a(x(t)), \quad t = (t^\alpha) \in \Omega_{0\tau}, \quad a = \overline{1, n-1}, \quad \alpha = 1, \dots, m. \quad (PDE)$$

This sort of controlled PDE is called a *driftless control system*. Of course, the piecewise complete integrability conditions

$$\left(\frac{\partial u_\alpha^a}{\partial t^\beta} - \frac{\partial u_\beta^a}{\partial t^\alpha} \right) X_a = u_\alpha^a u_\beta^b [X_a, X_b] \quad (CIC)$$

restrict the controls, excepting the case when they are identically satisfied.

To show that the driftless control system is multitime controllable, by bang-bang controls (see also [10]), we use the next *multitime minimum problems*

Case of multiple integral functional Let $U = [-1, 1]^{m(n-1)} \subset R^{m(n-1)}$ be the control set. Giving the starting point $x_0 \in R^n$, find an optimal control $u^*(\cdot)$ such that

$$I(u^*(\cdot)) = \min_{u(\cdot)} \int_{\Omega_{0\tau}} dt^1 \dots dt^m,$$

using a completely integrable two-time evolution (PDE) as constraint and supposing that (CIC) are identically satisfied. Since $\tau^{*1} \dots \tau^{*2} = I(u^*(\cdot))$, the optimal point $\tau^* = (\tau^{*1}, \dots, \tau^{*m})$ ensures the minimum multitime "volume" to steer to the origin. This two-time optimum problem consists in devising a control such that to transfer a given initial state to a specified target (controllability problem).

Solution We apply the multitime maximum principle which proves the existence of a bang-bang control. The Hamiltonian $H(x, p, u) = -1 + p_i^\alpha X_a^i(x) u_\alpha^a$ gives the adjoint PDE system $\frac{\partial p_i^\alpha}{\partial t^\alpha}(t) = -p_i^\alpha(t) \frac{\partial X_a^i}{\partial x^j}(x(t)) u_\alpha^a(t)$. The extremum of the linear function $u \rightarrow H$ exists since the set U is compact; for optimum, the control vectors $u_\alpha = (u_\alpha^1, \dots, u_\alpha^{n-1})$ must be vertices of ∂U . If $Q_a^\alpha(t) = p_i^\alpha(t) X_a^i(x(t))$ are the switching functions, then each optimal control u_α^{*a} is of the form

$$u_\alpha^{*a} = -\text{sign}(Q_a^\alpha(t)) = \begin{cases} 1 & \text{for } Q_a^\alpha(t) < 0 : \text{bang-bang control} \\ \text{undetermined} & \text{for } Q_a^\alpha(t) = 0 : \text{singular control} \\ -1 & \text{for } Q_a^\alpha(t) > 0 : \text{bang-bang control.} \end{cases}$$

Suppose the Lebesgue measure of each set $\{t \in \Omega_{0\tau} : Q_a^\alpha(t) = 0\}$ vanishes. Then the singular control is ruled out and the remaining possibilities are bang-bang controls. This optimal control is discontinuous since each component jumps from a minimum to a maximum and vice versa in response to

each change in the sign of each $Q_a^\alpha(t)$. The piecewise complete integrability identities keep only the control vectors (vertices of ∂U) u_α which satisfy $u_\alpha = \pm u_1$. Each optimal m -sheet $x(t)$ is a *soliton* solution.

Case of curvilinear integral functional

Optimal terminal value Let $U = [-1, 1]^{m(n-1)} \subset R^{m(n-1)}$ be the control set. Suppose we have to

Minimize the terminal cost functional

$$Q(u(\cdot)) = x^n(\tau)$$

subject to the driftless control system

$$\frac{\partial x}{\partial t^\alpha}(t) = u_\alpha^a(t)X_a(x(t)), \quad t = (t^\alpha) \in \Omega_{0\tau}, \quad x(0) = x_0, \quad a = \overline{1, n-1}, \quad \alpha = 1, \dots, m. \quad (PDE)$$

Solution Since the control Hamiltonian $H(x, p, u) = p_i X_a^i(x) u^a$ is linear in the control, the optimal control is a bang-bang. Automatically we find the optimal costate function and the optimal evolution.

6 Optimal control problems on Tzitzeica surfaces

Let R_+^2 be endowed with the product order. Let $\Omega \subset R_+^2$ be the bi-dimensional interval determined by the opposite diagonal points $(0, 0)$ and (u^1, v^1) .

Problem: *find*

$$\max_h \int_\Omega L(u, v, \vec{r}(u, v), h(u, v)) du dv$$

constrained by (non-ruled Tzitzeica surfaces)

$$\vec{r}_{uu} = \frac{h_u}{h} \vec{r}_u + \frac{1}{h} \vec{r}_v, \quad \vec{r}_{uv} = h \vec{r}, \quad \vec{r}_{vv} = \frac{1}{h} \vec{r}_u + \frac{h_v}{h} \vec{r}_v, \quad (\ln h)_{uv} = h - \frac{1}{h^2},$$

$$\vec{r}(0, 0) = \vec{r}_0, \quad \vec{r}(u^1, v^1) = \vec{r}_1,$$

where $h(u, v)$ is the control.

To solve the problem we use

$$\mathcal{L} = L + \langle \vec{a}, \frac{h_u}{h} \vec{r}_u + \frac{1}{h} \vec{r}_v - \vec{r}_{uu} \rangle + \langle \vec{b}, h \vec{r} - \vec{r}_{uv} \rangle$$

$$+\langle \vec{c}, \frac{1}{h} \vec{r}_u + \frac{h_v}{h} \vec{r}_v - \vec{r}_{vv} \rangle + d \left(h - \frac{1}{h^2} - (\ln h)_{uv} \right).$$

Explicitely, find

$$\max_h \int_{\Omega} (h^2(u, v) + ||\vec{r}(u, v)||^2) dudv.$$

7 Phytoplankton growth model

Open problem *Transform the next ODE systems into Pfaff systems and study their stochastic perturbations.*

Alessandro Abate, Ashish Tiwari, Shankar Sastry, Box Invariance for biologically-inspired dynamical systems

(i) O. Bernard and J.-L. Gouze, "Global qualitative description of a class of nonlinear dynamical systems," Artificial Intelligence, vol. 136, pp. 29-59, 2002:

Consider the following Phytoplankton Growth Model:

$$\dot{x}^1 = 1 - x^1 - \frac{1}{4} x^1 x^2, \quad \dot{x}^2 = 2x^2 x^3 - x^2, \quad \dot{x}^3 = \frac{1}{4} x^1 - 2x^3^2,$$

where x^1 denotes the substrate, x^2 the phytoplankton biomass, and x^3 the intracellular nutrient per biomass.

(ii) A. Julius, A. Halasz, V. Kumar, and G. Pappas, Controlling biological systems: the lactose regulation system of Escherichia Coli, in American Control Conference 2007:

The dynamics of tetracycline antibiotic in a bacteria which develops resistance to this drug (by turning on genes $TetA$ and $TetR$) can be described by the following hybrid system:

$$\begin{aligned} \dot{x}^1 &= f - \frac{1}{3} x^1 x^3 + \frac{1}{8000} x^2, \quad \dot{x}^2 = \frac{3}{200} u_0 - \frac{7}{2} x^3 x^4 \\ \dot{x}^3 &= \frac{1}{3} x^1 x^3 - \frac{1}{2500} x^2, \quad \dot{x}^4 = f - \frac{11}{40000} x^4, \end{aligned}$$

where $f = \frac{1}{2000}$ if $TetR > \frac{1}{50000}$ and $f = \frac{1}{40}$ otherwise (f is the transcription rate of genes, which are inhibited by $TetR$), and x^1, x^2, x^3, x^4 are the cytoplasmic concentrations of $TetR$ protein, the $TetR-Tc$ complex, Tetracycline, and $TetA$ protein, and u_0 is the extracellular concentration of Tetracycline.

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Prof. Dr. Constantin Udriște, University Politehnica of Bucharest, Faculty of Applied Sciences, Departament of Mathematics-Informatics, Splaiul Independentei 313, 060042 Bucharest, Romania,
E-mail: udriste@mathem.pub.ro, anet.udri@yahoo.com